Three Essays on Adverse Selection with Applications to Central Bank Policy

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Introduction

In an interbank market, banks trade credits at the interbank rate to adjust their cash holdings. The interbank credit conditions influence the credit conditions for households and firms. Therefore, because a central bank holds the instruments to steer the interbank rate, it is able to impact the economic development. If the central bank wants to foster economic growth, it lowers the interbank rate. To this end, the central bank uses reverse open market operations to offer credit to interbank borrowers.

There are two types of reverse open market operations, fixed and variable rate tenders. In a fixed rate tender, the central bank announces the interest rate for central bank credits in advance and banks bid the amount of credits they wish to get from the central bank. In a variable rate tender, banks bid both the interest rate they are willing to pay and the amount of central bank credits they wish to transact.

Central banks do not necessarily stick to one of the two tender types. For example, from 1999 to June 2000, the European Central Bank (ECB) conducted its reverse open market operations as fixed rate tenders. Because of low allotment ratios, the ECB started to use variable rate tenders and sticked to this procedure from June 2000 to October 2008. In October 2008, the ECB switched back to the fixed rate tender. The fact that both types of reverse open market operations are actually in use naturally raises the following question. How do fixed and variable rate tenders differ in their effect on the interbank market?

From a micro-theoretic point of view, the answer to this question builds on comparative statics analysis, i.e., the comparison of the market outcome before and after exogenous parameter changes. Comparative statics results in turn are easier to interpret if the market outcome is unique. This dissertation is a collection of three essays addressing these aspects of equilibrium analysis in markets with asymmetric information. The first essay examines the uniqueness of the market-clearing equilibrium and its refinements. The second essay explores how equilibrium price, trade volume, and excess supply in the unique equilibrium react to exogenous parameter changes. The third essay picks up the motivating example. It answers the question whether and how a fixed rate tender differs from a variable rate tender in its effect on the interbank market.

Chapter 1 delivers conditions for the uniqueness of the market-clearing equilibrium and its possible refinements in markets with asymmetric information on the good's quality. In these markets, sellers observe the quality of their good, whereas buyers only observe the average quality of goods in the market, but not the individual quality of a good.

The chapter is divided into three parts, each of them discussing a specific equilibrium concept. The first part of the analysis considers the market-clearing equilibrium. Multiple market-clearing equilibria may arise if not only supply, but also demand is increasing in price. This in turn requires that an increase in price leads to a strong inflow of sellers with high-quality goods. As a result, buyers find it more attractive to buy at higher than at lower prices. It is shown that a non-increasing elasticity of supply is sufficient for demand to be weakly decreasing in price and hence, the market-clearing equilibrium to be unique. Thus, it suffices that the inflow of sellers with high-quality goods decreases with price. Most standard quality distributions satisfy this requirement.

The second part of the analysis studies a refinement of the market-clearing equilibrium. This refinement accounts for the possibility that the market-clearing price does not necessarily maximize individual utility. Hence, instead of trading at the market-clearing price, buyers and sellers may want to renegotiate the trading price after being matched. However, if the elasticity of supply is not only non-increasing but also sufficiently low, agents always unilaterally offer to trade at the unique market-clearing price.

The third part of the analysis addresses the possibility of non-clearing equilibria. It studies the conditions under which each type of agent unilaterally offers to trade at a unique price, but this price is not the market-clearing price. Instead, there

is either excess demand or excess supply. The analysis shows that there are only equilibria with excess supply. For such an equilibrium to be unique, it suffices to impose a constant elasticity of supply.
The main result of Chapter 1 is that the uniqueness of equilibrium in markets with asymmetric information can be achieved by imposing restrictions on the elasticity of supply.

Chapter 2 builds on the equilibrium characterization in Chapter 1 and presents comparative statics analysis with respect to changes in the type distributions of market participants. In particular, it provides sufficient conditions for monotone comparative statics of equilibrium price, trade volume, and excess supply in markets with asymmetric information. If there is asymmetric information on quality, the buyers' demand not only depends on the price, but also on the average quality in the market. Thus, exogenous changes in the distribution of quality not only affect supply, they also affect demand.
The chapter contains three parts. The first part of Chapter 2 examines exogenous changes in the mix of market participants which unambiguously decrease equilibrium trade volume. It builds on the observation that for equilibrium trade volume to decrease, it suffices that both demand and supply react negatively to the exogenous changes. Thus, the exogenous changes have to be such that the quality distribution first-order stochastically dominates the initial one, and the elasticity of supply and the aggregate willingness to buy decrease.
The second part of Chapter 2 studies comparative statics of the equilibrium price. Monotone predictions of prices require a different set of conditions on the exogenous changes than predictions of trade volume. For the equilibrium price to increase, it suffices that the demand reaction to exogenous changes is positive, whereas the supply reaction is negative. Thus, for a higher equilibrium price, the quality distribution has to change such that it first-order stochastically dominates the initial one, and the elasticity of supply and the aggregate willingness to buy are required to increase.
The last part of Chapter 2 presents sufficient conditions for an unambiguous increase in equilibrium excess supply. Since excess supply denotes the difference between equilibrium supply and equilibrium demand, an increase in equilibrium supply and

a decrease in equilibrium demand suffice to raise equilibrium excess supply. The conditions for unambiguous predictions of excess supply relate to the conditions for trade volume. In particular, higher excess supply requires that both the elasticity of supply and the aggregate willingness to buy decrease. However, the quality distribution has to change such that it is first-order stochastically dominated by the initial one.
The main result of Chapter 2 is that for unambiguous predictions of the market outcome, exogenous changes in the type distributions of market participants need to fulfill well-defined conditions. There are different sets of conditions for equilibrium price, trade volume, and excess supply.

Chapter 3 builds on the methods and results in Chapter 1 and 2. In particular, Chapter 3 studies fixed and variable rate tenders and how the interbank market reacts to these tenders. It is assumed that in the interbank market, lenders cannot observe the borrowers' individual default probabilities. Thus, all credits are settled at the same interest rate. This raises the problem of adverse selection. Because borrowers with a high probability of default have a high willingness to pay, an interest rate only attracts borrowers with relatively high default probabilities. As a result, lenders may not be willing to lend at lower rates and the interbank rate may be high. This in turn may lead to a central bank intervention.
Chapter 3 is divided into two parts. The first part shows that both the fixed and the variable rate tender decrease the unique market-clearing interbank rate by lowering demand for interbank credits. The differences between the two tenders are that the variable rate tender may lower the interbank market by more than the fixed rate tender, whereas the fixed rate tender allows the central bank to choose from a set of possible interbank rates.
The second part of the analysis reveals that by lowering the interbank rate, both the fixed and the variable rate tender increase social welfare, i.e., the sum of aggregate rents of borrowers, lenders, and the central bank. However, the tenders differ insofar as the fixed rate tender increases the aggregate welfare of borrowers and lenders by more than the variable rate tender. In contrast, the central bank's rent is higher with the variable rate tender than with the fixed rate tender.
The main result of Chapter 3 is that with asymmetric information on default prob-

abilities, fixed and variable rate tenders differ in their effect on the interbank rate and welfare.

The three essays shed light on equilibrium analysis and central bank interventions in markets with asymmetric information. The first two essays specify conditions for the uniqueness of equilibrium and unambiguous comparative statics results for equilibrium price, trade volume, and excess supply. These results and the underlying methods are useful to answer a wide range of economic questions. In particular, they allow discussing the motivating example. In the third essay it is shown that, given asymmetric information on default probabilities, fixed and variable rate tenders differ in their effect on the interbank market. The drop in the interbank rate and the increase in the central bank's rent are smaller, whereas the increase in aggregate welfare of borrowers and lenders is larger with a fixed than with a variable rate tender.

Chapter 1

Unique Market-Clearing Equilibrium in Markets for "Lemons"

joint with Christian Ewerhart

1 Introduction

In a market where the quality of the good differs across sellers, buyers may not be able to observe the individual quality of the good. In contrast to the sellers which are fully informed about the quality of their endowment, buyers may only observe the average quality of goods in the market. Because buyers cannot distinguish goods of different quality, all goods sell at the same price. Then, the buyer's decision to buy at this price depends on the average quality of goods offered for sale and her individual preference for quality. In contrast, the seller's decision whether to sell at this price depends on her reservation price, which is assumed to be increasing in quality. This, however, may lead to adverse selection, the so-called "lemons" problem. More precisely, a price only attracts "lemons", i.e., sellers with relatively low reservation prices and accordingly, relatively low quality. The standard example is the market for used cars. While sellers have enough time and knowledge to judge the quality of a used car, buyers are restricted in their ability to learn the quality of this car before buying it.

If buyers and sellers in this market are price-takers, the equilibrium concept of market-clearing seems suitable to describe the market outcome. However, in markets with asymmetric information, the market-clearing equilibrium features some remarkable characteristics. First, there may be no market-clearing equilibrium at all. Akerlof (1970) presents an example where average quality at each price is so low that buyers never find it optimal to buy goods. Second, there may be more than one market-clearing price and some of them may not be stable equilibria (cf. Wilson, 1979). This may occur if an increase in price leads to a strong increase in average quality such that demand for goods is actually increasing in price. If buyers and sellers had the choice, they would then prefer the higher over the lower market-clearing price. Third, the market-clearing price at which goods are traded does not necessarily maximize all sellers' and buyers' utilities. While sellers generally prefer the highest possible price, the most favorable price may differ among buyers if they differ in quality preferences. Hence, if buyers or sellers had some price-setting power, trading prices may differ from the market-clearing price. These characteristics make the study of markets with asymmetric information more difficult than markets with symmetric information.

In this paper, we take a closer look at the specific characteristics of the market-clearing equilibrium and present conditions for a simple equilibrium analysis in markets with asymmetric information. In particular, we discuss the market-clearing concept and its possible refinements and study equilibrium uniqueness. Thereby, we use the standard set-up as introduced by Akerlof (1970) and Wilson (1980). We show that a non-increasing elasticity of supply is sufficient for a unique market-clearing equilibrium. The condition on the elasticity of supply can be reformulated as the requirement that the distribution of quality q has to be log-concave in $\log q$. We then show that many quality distributions satisfy this requirement. The uniqueness condition also carries over to a set-up where sellers offering the same quality may have different reservation prices.

In our *standard* set-up, buyers and sellers are randomly matched to trade at the market-clearing price. However, it may be optimal for the pair to renegotiate the trading price after being matched to maximize individual utility. We therefore introduce an *adjusted* set-up which allows a single seller or a single buyer to unilaterally announce a trading price. In this adjusted set-up, the equilibrium is defined as a refined market-clearing equilibrium where neither buyers nor sellers have an incentive to unilaterally announce a price different from the market-clearing price. We show that for the unique market-clearing equilibrium to be a refined market-clearing equilibrium, the elasticity of supply not only has to be non-increasing in price, it also has to be sufficiently low at high prices. We extend our analysis by incorporating the possibility that goods are not necessarily traded at a market-clearing price. We define a refined equilibrium as some price which implies either excess demand or excess supply and neither buyers nor sellers have an incentive to unilaterally announce another price. We find that there is no refined equilibrium with excess demand, but there may exist a refined equilibrium with excess supply. We show that for the existence and uniqueness of the refined equilibrium with excess supply, it suffices to impose a constant elasticity of supply.[1]

The rest of the paper is organized as follows. Section 2 discusses related literature. In Section 3, we outline the model. In Section 4, we derive sufficient conditions for a unique market-clearing equilibrium. Section 5 shows that many distributions feature a unique market-clearing equilibrium. Section 6 presents sufficient conditions for a unique refined market-clearing equilibrium. We study the uniqueness of a refined equilibrium in Section 7. In Section 8, we extend the model by introducing multidimensional seller types. Section 9 concludes.

2 Related literature

Most closely related is the contribution of Wilson (1980). He uses the standard set-up to derive a condition for multiple market-clearing equilibria. This condition requires the slope of average quality as a function of price to be large enough. The difference here is that our condition on the elasticity of supply can be reformulated as a restriction on the primitive. Moreover, Wilson (1980) also studies the uniqueness of more refined market-clearing concepts where either buyers or sellers have price-setting power. That is, every buyer simultaneously announces a price and sellers search for the most favorable price and vice versa. Besides allowing all buyers or sellers to make simultaneous announcements, Wilson's (1980) refined concepts dif-

[1]Some of the results in this paper have appeared in a preliminary form in our IEW Working Paper No. 455.

fer from ours insofar as they require strong assumptions on expectations of market participants on the price-setting behavior of others.

Rewriting Wilson's (1980) condition for a unique market-clearing equilibrium, Rose (1993) finds that an elasticity of average quality smaller than one is sufficient for uniqueness. Considering this condition as analytically intractable, he shows numerically that many standard distributions fulfill it. Again, our uniqueness condition is more informative because it restricts the primitive. Moreover, reformulating our uniqueness condition as described above allows to reproduce and extend Rose's numerical results using simple calculus.

Several other papers have delivered conditions for a unique market-clearing equilibrium in markets with asymmetric information. However, the set-ups used in these studies differ from ours and therefore, from the standard set-up. In contrast to our set-up where there is a continuum of buyer types with individual preference for quality, Bagnoli and Bergström (2005) study an example with a single buyer type.[2] They show that in their set-up, there is a unique market-clearing equilibrium if the cumulative distribution function of quality is log-concave in quality.[3] Bigelow (1990) studies the unique market-clearing equilibrium in a market with a single buyer and a single seller and two possible realizations of quality. He finds a uniqueness condition which restricts the allowable convexity of the quality distribution.

Our study also refers to the vast literature on rationing, i.e., on excess demand or supply, especially in the context of credit markets. In these markets, the uniqueness of equilibrium hinges on the assumption that the lender's return on a loan as a function of the interest rate has a unique maximum. Stiglitz and Weiss (1981) argue that this return function has an interior maximum because the adverse selection effect dominates the direct effect of an increase in the interest rate at high rates. However, Arnold and Riley (2009) show that there is no such interior maximum. Instead, the return on a loan is maximized at a very high interest rate where only the most risky borrowers are left in the market and there are no infra-marginal borrowers which extract informational rents. The return maximization argument also applies to efficiency wages in markets with asymmetric information on individual

[2]Their study was first published in 1989.

[3]Note that our restriction on the quality distribution, i.e., $\partial \log F(q)/\partial \log q$, can be rewritten as $q \cdot (\partial \log F(q)/\partial q)$.

productivity of employees.[4] A comprehensive analysis of the labor market's unique equilibrium with rationing can be found in Stiglitz (1976), for example.
With the literature on monopolistic power, search, and matching, this paper shares the property that one single agent may have some price-setting power. Examples of markets with a single price-setting seller are given by Adriani and Deidda (2009), Bester and Ritzberger (2001), and Ellingsen (1997). In search and matching models, trading partners meet pairwise either by random matching or by searching at some cost. Price is bilaterally negotiated or determined by take-it-or-leave-it offers (see, e.g., Inderst and Müller, 2002, Bester, 1988, Guerrieri, Shimer, and Wright, 2010, Ponsatí and Sákovics, 2008, and Blouin, 2003).

3 Set-up

In this section, we introduce the set-up and define the market-clearing and the refined market-clearing equilibrium.

3.1 Price-taking buyers and sellers

There is a market for an indivisible good. Assume for the moment that in this market, both sellers and buyers are price-takers. We will later relax the price-taking assumption.
In this market, there is a continuum of *sellers*, whose population size is normalized to one. Each seller possesses one unit of the good and wants to sell it in the market. Endowments across sellers differ in quality q, where $q \in [q_0, q_1]$ with $q_0 < q_1$. The quality of endowments is distributed according to some strictly positive density f on $[q_0, q_1]$. We assume that q not only describes the quality of the seller's endowment, but also perfectly describes the seller's reservation price. Hence, a seller is willing to sell her endowment if the offered price p is weakly higher than quality q. It follows that if there is a single market price p, supply at this price is given by $S(p) = F(p)$, where F denotes the cumulative distribution function of q.
There is also a continuum of *buyers*. Again, we normalize the size of the buyer population to one. Buyers have no endowment and want to buy one unit of the

[4] See Akerlof and Yellen (1986) for an overview of the early literature.

good. In contrast to sellers, buyers cannot observe the individual quality of a unit. However, they are able to infer average quality of units on offer from price p. It is assumed that buyers differ in their preference for quality, i.e., each buyer has an individual preference type $t \in [t_0, t_1]$ with $t_0 < t_1$ and $t_1 > 1$. Preference types are distributed according to some density h on $[t_0, t_1]$. Note that the individual preference type is private information. A buyer is willing to buy a good if $t\bar{q}(p) \geq p$, where $\bar{q}(p) = E[q \mid q \leq p]$ is the average quality of units on offer. Let $\bar{q}(p) = q_0$ at price $p = q_0$. Consequently, if p is the single market price, demand at this price is given by $D(p) = 1 - H(p/\bar{q}(p))$, where $H(t)$ is the fraction of buyers with preference type $\leq t$.

Having introduced the market characteristics as well as demand and supply, we now turn to the market equilibrium. Taking into account that both sellers and buyers are price-takers, we define a market-clearing equilibrium as follows.

Definition 1. A market-clearing equilibrium is a price p^* for which $S(p^*) = D(p^*)$, i.e.,

$$F(p^*) = 1 - H(p^*/\bar{q}(p^*)). \tag{1}$$

Intuitively, price-taking agents decide at which prices they are willing to buy or sell, respectively. Given the resulting demand and supply functions, the market-clearing price p^* can be determined. Then, unless there is no central clearing institution which carries out all trades, buyers willing to buy at this price are randomly matched with sellers willing to sell at p^* and units are traded bilaterally at price p^*.

3.2 A single agent with price-setting power

Even though any individual price-setting attempts are excluded in Definition 1, bilateral matching leaves room for buyers and sellers to take a more active role in the price-setting mechanism. In particular, buyers and sellers may find it optimal to renegotiate the trading price as soon as they are matched. To be able to discuss price renegotiations and the conditions under which the result of these renegotiations is p^*, we adjust the set-up introduced in Section 3.1 and refine the definition of equilibrium.

Suppose that a single buyer is given some price-setting power. In this case, the market works as follows.

Definition 2 (Buyer announcement). Assume that there is a market-clearing price p^* for which $D(p^*) = S(p^*)$. Before buyers are matched with sellers and any trade takes place, one single buyer who is willing to buy at p^*, i.e., a buyer with $t \in [p^*/\bar{q}(p^*), t_1]$, is chosen at random. This buyer has the possibility to announce the price at which she prefers to buy. The announced price may differ from the market-clearing price p^*. However, if the buyer is indifferent between the market-clearing price p^* and any other price p, the buyer sticks to p^*. We assume that all sellers can costlessly compare the announced price p with the market-clearing price p^*. Moreover, we assume that all sellers who prefer p over p^* are equally likely to sell their good at p. Those sellers which cannot sell at the announced price may sell their endowment at p^*.

Suppose now that instead of a single buyer a single seller is given some price-setting power. In this case, the market works as follows.

Definition 3 (Seller announcement). Assume that there is a market-clearing price p^* for which $S(p^*) = D(p^*)$. Before sellers and buyers are matched and goods are traded, a single seller willing to sell at p^*, i.e., a seller with $q \leq p^*$, is randomly chosen. This seller has the possibility to announce the price at which she prefers to sell. The announced price may differ from p^*. However, if the seller is indifferent between p^* and any other price, she will announce p^*. We assume that all buyers can costlessly compare the announced price p with the market-clearing price p^*. Moreover, we assume that buyers expect the unit quality of the single seller announcing p to be $\bar{q}(p)$ for $p \leq p^*$ and $\bar{q}(p^*)$ for $p > p^*$. All buyers who prefer p over p^* are equally likely to buy the good at p. Those buyers who cannot buy at p may buy a good at p^*.

If all buyers with $t \in [p^*/\bar{q}(p^*), t_1]$ and all sellers with $q \leq p^*$ unilaterally announce the market-clearing price p^*, this price is robust to any possible buyer and seller announcement. Lemma 1 presents the condition for robustness of the market-clearing price.

Lemma 1. The market-clearing price p^* is robust to any unilateral announcement of buyers with $t \in [p^*/\bar{q}(p^*), t_1]$ and sellers with $q \leq p^*$ if

$$p^* = \arg\max_p u(p; p^*, t), \tag{2}$$

where

$$u(p;p^*,t) = \begin{cases} 0 \quad \text{for } \; p < p^* \\ t\bar{q}(p) - p \quad \text{for } \; p \geq p^* \end{cases} \tag{3}$$

for $t \in [p^*/\bar{q}(p^*), t_1]$.

Proof. Assume the market-clearing price to be p^* and assume that there is a buyer announcement as defined in Definition 2. The buyer with $t \in [p^*/\bar{q}(p^*), t_1]$ may consider to unilaterally announce $p < p^*$. However, no seller is willing to offer her good at p because all sellers willing to sell at p^* are able to sell at this price and sellers always prefer higher over lower prices. Hence, because the buyer's utility is zero at p and non-negative at p^*, she rather sticks to p^* than announcing a price $p < p^*$. However, the buyer may also consider to announce $p > p^*$. At such a price, all sellers with $q \leq p$ also want to sell at p. The buyer's utility at p is therefore given by $t\bar{q}(p) - p$. Thus, the buyer's utility from unilateral announcement p is given by $u(p;p^*,t)$ as defined in Lemma 1. If p^* maximizes $u(p;p^*,t)$ for $t \in [p^*/\bar{q}(p^*), t_1]$, every buyer with such a type will unilaterally announce p^* such that p^* is robust to any buyer announcement.
Consider now the case where instead of a buyer announcement there is a seller announcement as defined in Definition 3. The seller with $q \leq p^*$ has no incentive to announce a price $p < p^*$ because she has always the possibility to sell her good at the market-clearing price p^*. However, the seller may consider to unilaterally announce a price $p > p^*$. Such an announcement generates positive utility with $p - q > p^* - q$, if there are some buyers willing to trade at price p. If a seller announces p, a buyer's utility from buying at $p > p^*$ is given by $t\bar{q}(p^*) - p$ for $t \in [t_0, t_1]$. Because $t\bar{q}(p^*) - p < t\bar{q}(p^*) - p^*$ for $p > p^*$, no buyer is willing to offer to buy at $p > p^*$. Hence, the seller's utility from announcing $p > p^*$ is zero such that she will unilaterally announce p^*. As a result, p^* is robust to any seller announcement. □

In this adjusted set-up, the equilibrium is called the refined market-clearing equilibrium and it is defined as follows.

Definition 4. A refined market-clearing equilibrium is a price p_r^* which satisfies the following two conditions:

i) $S(p_r^*) = D(p_r^*)$, i.e., $F(p_r^*) = 1 - H(p_r^*/\bar{q}(p_r^*))$

ii) p_r^* is robust to any unilateral announcement of buyers with $t \in [p_r^*/\bar{q}(p_r^*), t_1]$ and sellers with $q \leq p_r^*$.

We now discuss the conditions for a unique market-clearing and refined market-clearing equilibrium.

4 Unique market-clearing equilibrium

Clearly, a unique market-clearing equilibrium requires supply and demand as functions of price to intersect at most once. Since $f > 0$, supply $S(p) = F(p)$ is continuous and strictly increasing on the price interval $[q_0, q_1]$, while constant and strictly positive for $p > q_1$, and zero for $p \leq q_0$. Demand for the indivisible good depends on the price-quality ratio $p/\bar{q}(p)$. Average quality is not defined for $p < q_0$. Moreover, it is increasing on the price interval $[q_0, q_1]$, and constant and positive for $p > q_1$. Therefore, demand is not defined for prices $p < q_0$ such that it cannot cross supply at these prices. Moreover, demand is strictly declining for prices $p > q_1$. At price $p = q_0$, the price-quality ratio equals one. Given that $t_1 > 1$, demand is positive, whereas supply is zero. Hence, for at most one market-clearing equilibrium it suffices that demand is weakly declining on the price interval $[q_0, q_1]$. This in turn requires that for prices $[q_0, q_1]$, the price-quality ratio is weakly increasing in price. Define the price elasticity of supply as $\varepsilon^S = (\partial S/\partial p) \cdot (p/S)$. As we will show below, a weakly declining elasticity of supply is sufficient for a weakly increasing price-quality ratio. Theorem 1 formalizes these results.

Theorem 1. Assume $\partial \varepsilon^S/\partial p \leq 0$ for prices $p \in [q_0, q_1]$. Then $\partial D/\partial p \leq 0$ for $p \geq q_0$, and the market-clearing equilibrium is unique.

Proof. For demand $D(p) = 1 - H(p/\bar{q}(p))$ to be weakly decreasing on the price interval $[q_0, q_1]$, it suffices to show that $p/\bar{q}(p)$ weakly increases in p, i.e.,

$$\frac{\partial}{\partial p} \frac{p}{\bar{q}(p)} = p \left(\frac{\bar{q}(p)}{p} - \frac{\partial \bar{q}(p)}{\partial p} \right) \geq 0 \tag{4}$$

for all $p \in [q_0, q_1]$. Applying Van den Berg (1994), the condition $\partial\varepsilon^S/\partial p \leq 0$ suffices for $\mathrm{d}\log\bar{q}(p)/\mathrm{d}\log p \leq 1$ to be satisfied (the details are left to the appendix). Using the fact that $\mathrm{d}\log p = (1/p)\mathrm{d}p$, one can rewrite $\mathrm{d}\log\bar{q}(p)/\mathrm{d}\log p \leq 1$ as $\mathrm{d}\bar{q}(p)/\mathrm{d}p \leq \bar{q}(p)/p$. This implies that with a non-increasing elasticity of supply, the bracket in (4) is non-negative and the weak inequality is satisfied. It follows that under a non-increasing elasticity of supply, demand $D(p)$ is weakly declining on the interval $[q_0, q_1]$, and strictly declining or zero for $p > q_1$. Hence, $D(p) - S(p)$ is strictly decreasing or negative for all $p \geq q_0$ and there is at most one market-clearing equilibrium. □

Intuitively, to get a declining demand function, the price-quality ratio $p/\bar{q}(p)$ has to get worse with price. Consider the marginal buyer type which is indifferent between buying and dropping out of the market at price p. Since the marginal buyer's type is given by $t = p/\bar{q}(p)$, demand is decreasing in price if the price-quality ratio is increasing. This in turn requires the increase in average quality to decrease with price. I.e., the inflow of sellers with high quality goods needs to decrease with price. A weakly decreasing elasticity of supply reflects exactly this requirement.
Figure I illustrates an example for a market with a unique market-clearing equilibrium. Thereby, quality q is log-normally distributed on $[0, 3.4]$, whereas t is uniformly distributed on $[1, 2.2]$.

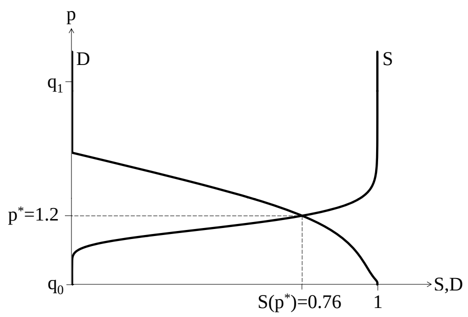

Figure I: Example for a unique market-clearing equilibrium

Because the distribution of quality is log-normal, the elasticity of supply is decreasing and the price-quality ratio is increasing for prices $p \in [q_0, q_1]$. This leads to an everywhere weakly declining demand and a unique market-clearing equilibrium.

Wilson (1979) presents an example where in our set-up, the market-clearing equilibrium is not unique. As the left-hand side of Figure II illustrates, there are three market-clearing equilibria in his example.

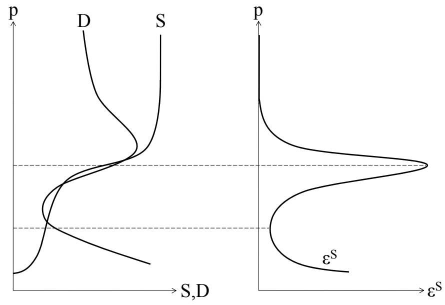

Figure II: Multiplicity of the market-clearing equilibrium in an example due to Wilson (1979)

We reconstructed Wilson's example and derived the corresponding elasticity of supply, as shown on the right-hand side of Figure II. It turns out that for the relevant prices, $\partial\varepsilon^S/\partial p \leq 0$ is not satisfied. Instead, the elasticity of supply *increases* at intermediate price levels (between the two dashed lines). At these prices, there is a strong inflow of high quality goods sellers such that the price-quality ratio decreases. As a result, demand increases at intermediate price levels and there are multiple market-clearing equilibria. Wilson's example also illustrates that demand may continue to increase when the elasticity of supply already begins to decline. Clearly, this results from the fact that $\bar{q}(p)$ averages over *all* submarginal qualities.

5 Distributions implying uniqueness

In this section, we show that a wide range of quality distributions imply a unique market-clearing equilibrium.

To investigate the question which quality distribution features a non-increasing elasticity of supply and implies a unique market-clearing equilibrium, we reformulate the condition on the elasticity of supply in Theorem 1.

Theorem 2. Assume that $F(q)$ is log-concave in $\log q$ for $q \in [q_0, q_1]$. Then, $\partial\varepsilon^S/\partial p \leq 0$ for $p \in [q_0, q_1]$.

Proof. Using $\partial p = p \cdot \partial \log p$ one can write

$$\frac{\partial\varepsilon^S(p)}{\partial p} = \frac{\partial}{\partial p}\left[\frac{\partial \log F}{\partial \log p}\right] = \frac{1}{p}\left[\frac{\partial^2 \log F}{\partial(\log p)^2}\right] \leq 0. \tag{5}$$

□

Since the closed-form representation of the cumulative distribution function may not always exist (see, for example, Ewerhart, 2011), we rewrite the condition in Theorem 2 as a condition on the density function.

Theorem 3. Assume that $f(q)$ is continuously differentiable and log-concave in $\log q$.[5] Then, $F(q)$ is log-concave in $\log q$.

Proof. Assume that $f(q)$ is log-concave in $\log q$. Write $\pi = \log q$. Then, $q = \exp(\pi)$, and $\log f(q) = \log f(\exp(\pi))$ is concave in π. Write $g(\pi) = f(\exp(\pi))\exp(\pi)$. From $\log g(\pi) = \log f(\exp(\pi)) + \pi$, it follows that $\log g(\pi)$ is concave in π. Hence, $g(\pi)$ is log-concave in π. By Theorem 1 in Bagnoli and Bergström (2005), the indefinite integral $G(\pi) = F(\exp(\pi))$ is therefore also log-concave in π. Thus, $F(q)$ is log-concave in $\log q$. □

The conditions in Theorem 2 and Theorem 3 allow to test whether a specific quality distribution implies a unique market-clearing equilibrium by using simple calculus. For illustration, consider the following example.

Example 1. Since $\partial \log q = (1/q) \cdot \partial q$, f is log-concave in $\log q$ if $q \cdot (\partial g(p)/\partial q) \leq 0$, where $g(q) = q \cdot (\partial \log f/\partial q)$. Assume that quality follows the Chi distribution such that

$$\log f(q) = (1 - \frac{k}{2})\log 2 + (k-1)\log q - \frac{q^2}{2} - \log\left(\int_0^\infty e^{-u}u^{k/2-1}du\right) \tag{6}$$

for $k = 1, 2, ...$ and $q \in [0, \infty]$. Then, $g(q) = k - 1 - q^2$ and

$$q \cdot \frac{\partial g(p)}{\partial q} = -2q^2 < 0. \tag{7}$$

[5] Note this assumption amounts to $qf(q)f''(q) + f'(q)f(q) - qf'(q)^2 \leq 0$ when f is smooth.

Thus, the Chi distribution features a non-increasing elasticity of supply.

So far, as to the best of our knowledge, the only study to test for uniqueness of the market-clearing equilibrium in the standard set-up for markets with asymmetric information is Rose (1993), which offers a numerical investigation of quality distributions. He starts from the observation that demand is decreasing in price if and only if the *elasticity of average quality* $\varepsilon = (\partial \overline{q}(p)/\partial p) \cdot (p/\overline{q}(p))$ is smaller than one. I.e.,

$$\varepsilon = \frac{p/\tilde{t} \cdot f(p/\tilde{t})}{\int_{q_0}^{p/\tilde{t}} f(q)dq} \left[\frac{p/\tilde{t}}{\overline{q}(p)} - 1 \right] < 1, \tag{8}$$

where $f(q)$ is the density of quality, $q \in [q_0, q_1]$, and $\tilde{t}$ denotes the sellers' common valuation of quality. Using numerical techniques, he shows that for most standard distributions, the market-clearing equilibrium is indeed unique.[6]

Applying Theorem 2 and Theorem 3, we are able to analytically reproduce all the numerical results in Rose (1993). Moreover, we are able to show that the Pareto, the Maxwell, and the power distribution feature a non-increasing elasticity of supply. It is also straightforward to apply Theorem 3 to any other continuously differentiable density function.

Theorem 2 and 3 relate to the results in Van den Berg (1994). He shows that if $1 - F(q)$ is log-concave in $\log q$, $\log q f(\log q)/(1 - F(\log q))$ is non-decreasing in $\log q$. Moreover, he finds that if $f(q)$ is log-concave in $\log q$, then, $1 - F(q)$ is log-concave in $\log q$. Comparing Van den Berg's (1994) result and ours, Theorem 3 should imply that if $F(q)$ is log-concave in $\log q$, $1 - F(q)$ is also log-concave in $\log q$. The corresponding proof will not be discussed in this paper.[7]

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[6] Rose's (1993) numerical analysis reveals that the gamma, chi squared, chi, exponential, log-normal, beta (for specific parameter values), uniform, half-normal, Rayleigh, Students' t, F-Ratio, and Weibull distributions feature a non-increasing elasticity of supply. For the normal, extreme-value, Cauchy, Laplace, and logistic distribution, however, the elasticity of supply is only non-increasing for higher values of q.

[7] Bagnoli and Bergström (2005) find that if $F(q)$ is log-concave in q, $1 - F(q)$ is also log-concave in q.

6 Refined market-clearing equilibrium

The discussion of the unique refined market-clearing equilibrium is twofold. On the one hand, it consists of the analysis of conditions under which only one of multiple market-clearing equilibria is also a refined market-clearing equilibrium. On the other hand, the uniqueness discussion also consists of the analysis of conditions under which a unique market-clearing equilibrium is also a refined market-clearing equilibrium. In this section, we will focus on the second part of the discussion. In particular, we study the conditions under which neither buyers nor sellers willing to trade at the unique market-clearing price p^* have an incentive to unilaterally announce a price $p \neq p^*$.
Suppose that the elasticity of supply is non-increasing and there is a unique market-clearing equilibrium with p^* for which $S(p^*) = D(p^*)$.

Theorem 4. Assume $\varepsilon^S \leq 1/t_1$ for $p \geq p^*$. Then, the unique market-clearing equilibrium p^* is also a refined market-clearing equilibrium p_r^*.

To prove Theorem 4, we need to show that the condition in Theorem 4 is sufficient for p^* to be robust to any unilateral announcement of buyers with $t \in [p^*/\bar{q}(p^*), t_1]$ and sellers with $q \leq p^*$, i.e., we need to determine the conditions under which Lemma 1 is satisfied.

Lemma 2. Assume $\varepsilon^S \leq 1/t_1$ for $p \geq p^*$. Then, $t\bar{q}(p^*) - p^* > t\bar{q}(p) - p$ for $t \in [t_0, t_1]$ and $p > p^*$.

Proof. Buyer t's utility is higher at p^* than at p if

$$\frac{\partial}{\partial p}\left(t\bar{q}(p) - p\right) = t\frac{\partial \bar{q}(p)}{\partial p} - 1 < 0 \tag{9}$$

for all $t \in [t_0, t_1]$ and $p \geq p^*$. Rewriting (9) by using

$$\bar{q}(p) = \frac{\int_{q_0}^{p} qf(q)dq}{F(p)} \tag{10}$$

yields

$$\frac{\partial \bar{q}(p)}{\partial p} = \frac{f(p)}{F(p)}\left(p - \bar{q}(p)\right) < \frac{1}{t}. \tag{11}$$

Thus, for buyer t's utility to be decreasing in price for $p \geq p^*$, it is sufficient to assume $f(p)p/F(p) = \varepsilon^S \leq 1/t$ for $p \geq p^*$. To make sure that the utility is decreasing for all types $t \in [t_0, t_1]$, $\varepsilon^S \leq 1/t_1$ is required. □

Intuitively, the utility of buyer t is decreasing in price if the decrease in utility caused by an increase in price is larger than the increase in utility caused by the increase in average quality. This in turn requires the increase in average quality and therefore, the inflow of high quality sellers to be relatively small. Note that from $t_1 > 1$ follows that $\varepsilon^S < 1$ for $p \geq p^*$.
If $\varepsilon^S \leq 1/t_1$ for $p > p^*$, p^* maximizes $u(p; p^*, t)$ as required in Lemma 1. But then, p^* is robust to any unilateral announcement of buyers with $t \in [p^*/\bar{q}(p^*), t_1]$ and sellers with $q \leq p^*$. Thus, the condition in Theorem 4 is indeed sufficient for the unique market-clearing equilibrium to be the refined market-clearing equilibrium.
Figure III illustrates an example for Theorem 4.

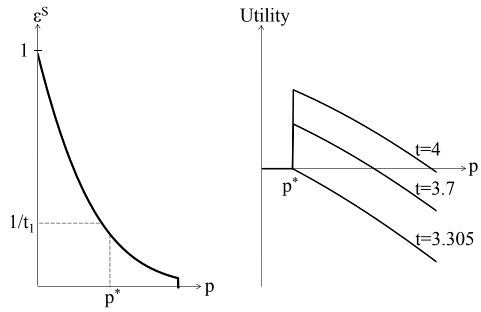

Figure III: Example for a unique refined market-clearing equilibrium

In the example, the size of the seller population equals one, whereas the size of the buyer population equals four. The density of quality is given by $f(q) = 1.2e^{-1.2q}$ with $q \in [0, 4]$ and the density of preference types is given by $h(t) = 1/3$ with $t \in [1, 4]$. The left-hand side of Figure III shows the elasticity of supply which is non-increasing in price for $p \in [q_0, q_1] = [0, 4]$ and lower than $1/t_1 = 1/4$ for $p \geq p^*$. Parameter values are chosen such that demand and supply intersect at price $p^* = 2.18$. The right-hand side of Figure III illustrates the utility of unilateral buyer announcement, i.e., $u(p; p^*, t)$, for types $t = 3.305, 3.7, 4$. Thereby, $t = 3.305$ is the marginal type at p^*. Thus, for all buyers with $t \in [p^*/\bar{q}(p^*), t_1] = [3.305, 4]$, utility of unilateral announcement is decreasing for $p \geq p^*$ and zero for $p < p^*$. As a result, $p^* = 2.18$ maximizes the buyers' utility from unilateral announcement and p^* is robust.

7 Extension: Refined equilibrium without market-clearing

Markets affected by adverse selection may not necessarily clear. There exists a large literature on non-clearing equilibria and rationing in these markets. Empirical evidence for the case where demand does not equal supply is delivered by King (2008), for example. He finds that banks with high default probability may attract less interbank loans than other banks such that these banks may not be able to fully cover their individual demand for loans. This empirical result indicates the presence of non-clearing in interbank markets.

In this section, we study the existence of a unique equilibrium price which is not necessarily the market-clearing price. We thereby use a variation of the refined market-clearing concept. More specifically, we analyze the conditions under which a unique price with excess demand or excess supply is robust to unilateral buyer and seller announcements.

7.1 Set-up

Now, the unique price we are interested in does not result from a market-clearing process. Instead, we assume that goods are supposed to be traded at a unique price p' which is the result of some price-setting mechanism. At this price, there may be either excess demand where there are more buyers than sellers willing to trade. Or there may be excess supply where there are more sellers than buyers willing to trade. To check whether the price p' is robust, we introduce both buyer and seller announcements as defined in Definitions 2 and 3, respectively. In these definitions, we thereby replace p^* by p' and $D(p^*) = S(p^*)$ by either $D(p') > S(p')$ for excess demand or $D(p') < S(p')$ for excess supply.

In this set-up, we call the equilibrium a refined equilibrium with either excess demand or excess supply and define it as follows.

Definition 5. A price p' is a refined equilibrium price p_e^* with excess demand if

1a) $D(p') > S(p')$, and

1b) p' is robust to any unilateral announcement of buyers with $t \in [p'/\bar{q}(p'), t_1]$ and sellers with $q \leq p'$.

Definition 6. A price p' is a refined equilibrium price p_e^* with excess supply if

2a) $D(p') < S(p')$ and

2b) p' is robust to any unilateral announcement of buyers with $t \in [p'/\bar{q}(p'), t_1]$ and sellers with $q \leq p'$.

7.2 Excess demand

Suppose that the price-setting mechanism results in a price p' where $D(p') > S(p')$ such that there is excess demand and there are more buyers willing to trade at p' than sellers.

Theorem 5. There is no refined equilibrium p_e^* with excess demand.

To prove Theorem 5, we need to show that there is no price p' with excess demand which is robust to unilateral buyer and seller announcement (see Definition 5).

Lemma 3. The price p' for which $D(p') > S(p')$ is robust to any unilateral announcement of buyers with $t \in [p'/\bar{q}(p'), t_1]$ and sellers with $q \leq p'$ if

$$p' = \arg\max_p u(p; p', t), \tag{12}$$

where

$$u(p; p', t) = \begin{cases} 0 \quad \text{for } \; p < p' \\ \pi t \bar{q}(p) - p \quad \text{for } \; p = p' \\ t\bar{q}(p) - p \quad \text{for } \; p > p' \end{cases} \tag{13}$$

for $t \in [t_0, t_1]$, and π is the probability of purchase at price p'. In addition, sellers with $q \leq p$ have to voluntarily forgo unilateral announcements $p > p'$.

Proof. Assume the trading price to be p' with $D(p') > S(p')$ and there is a buyer announcement as defined in Definition 2 (with the adjustments mentioned above).

The buyer with $t \in [p'/\bar{q}(p'), t_1]$ may consider to unilaterally announce $p < p'$. Because all sellers willing to trade at p' are able to sell their good at p', no seller will offer her good at a price $p < p'$. Thus, the buyer's utility from unilaterally announcing $p < p'$ is zero. If the buyer announces a price $p > p'$, all sellers willing to trade at p will offer their good at p. Therefore, the buyer's utility from unilaterally announcing $p > p'$ is given by $t\bar{q}(p) - p$. Finally, if the buyer decides to announce p', her utility is given by $\pi(t\bar{q}(p') - p')$, where $\pi < 1$. Thus, the buyer's utility from unilaterally announcing p is given by $u(p; p', t)$ for $t \in [p'/\bar{q}(p'), t_1]$ as defined in Lemma 3. If p' maximizes $u(p; p', t)$ for $t \in [p'/\bar{q}(p'), t_1]$, every buyer with such a type will unilaterally announce p' such that p' is robust to any buyer announcement. Consider now the case where instead of a buyer announcement there is a seller announcement as defined in Definition 3 (with the adjustments mentioned above). Because the seller with $q \leq p'$ is able to sell her good at price p', she has no incentive to announce any lower price $p < p'$. But the seller may consider to unilaterally announce a price $p > p'$. To see that there are buyers who offer to buy at $p > p'$, consider the following. A buyer's utility from buying at $p > p'$ is given by $t\bar{q}(p') - p$ for $t \in [t_0, t_1]$ and it is non-negative for buyers with type $t \geq p/\bar{q}(p') > p'/\bar{q}(p')$. Moreover, the probability of purchase at p', π, changes only marginally if there is a seller announcement p. Therefore, buyers with $t \geq p/\bar{q}(p')$ will always offer to buy at $p > p'$. This is because the additional opportunity to buy a good at p weakly increases expected utility. In case of offering to buy at p, the buyer is able to buy the good at p with some positive probability and there is still the possibility to buy the good at p' with probability π. In case of not offering to buy at p, she is only able to buy the good at p' with probability π. Thus, as long as there are buyers with $t \geq p/\bar{q}(p')$, the seller is able to seller her good at $p > p'$. Because $p - q > p' - q$ for any $q \leq p'$, a seller prefers p over p'. To definitely exclude the possibility that a seller announces $p > p'$, we need to assume that sellers with $q \leq p'$ voluntarily forgo to unilaterally announce $p > p'$. Then, the price p' is robust to any seller announcement. $\square$

Lemma 3 itself excludes the existence of a refined equilibrium with excess demand. Without further adjustments in the set-up, sellers will not voluntarily forgo to unilaterally announce $p > p'$. Thus, p' with $D(p') > S(p')$ is not robust to seller announcements. Moreover, a buyer with type $t > p'/\bar{q}(p')$ has always an incentive

to unilaterally announce a price $p > p'$ such that p' is also not robust to buyer announcements. To see why, consider the following.
A buyer with type $t > p'/\bar{q}(p')$ unilaterally announces a price $p + \epsilon > p'$ if

$$\pi(t\bar{q}(p') - p') < t\bar{q}(p' + \epsilon) - (p' + \epsilon) \tag{14}$$

for an arbitrarily small $\epsilon > 0$. Let $\epsilon \to 0$. Then, inequality (14) amounts to $\pi(t\bar{q}(p') - p') < t\bar{q}(p') - p'$. Because there is excess demand at p', $\pi < 1$ such that (14) is satisfied with strict inequality. But then, there always exists an arbitrarily small $\epsilon > 0$ such that (14) is satisfied and a buyer with $t > p'/\bar{q}(p')$ will have an incentive to unilaterally announce a price $p > p'$.
Intuitively, if there is excess demand at p', a buyer can increase the probability of purchase up to one by unilaterally announcing a slightly higher price $p > p'$. This increases the buyer's utility. Hence, any buyer with $t > p'/\bar{q}(p')$ will always announce some price $p > p'$ if she has the chance to do so. Moreover, to increase the probability of purchase, a buyer willing to pay $p > p'$ will always offer to buy at this price if there is a seller who unilaterally announces $p > p'$. Therefore, a unique refined equilibrium with excess demand cannot exist.

7.3 Excess supply

Consider now the case where the price-setting mechanism results in a price p' for which $D(p') < S(p')$ and there are more sellers in the market than buyers.

Theorem 6. Assume $\partial\varepsilon^S/\partial p = 0$ for $q_0 < p \leq q_1$. Then, $p' = q_1$ is the unique refined equilibrium p_e^* with excess supply.

To prove Theorem 6 we need to find conditions under which p' with $D(p') < S(p')$ is robust to any buyer and seller announcements (see Definition 6).

Lemma 4. The price p' for which $D(p') < S(p')$ is robust to any unilateral announcement of buyers with $t \in [p'/\bar{q}(p'), t_1]$ and sellers with $q \leq p'$ if

$$p' = \arg\max_p u(p; p', t), \tag{15}$$

where

$$u(p; p', t) = t\bar{q}(p) - p \tag{16}$$

for $t \in [t_0, t_1]$ and $p \geq q_0$.

Proof. Assume the trading price to be p' with $D(p') < S(p')$ and there is a buyer announcement as defined in Definition 2 (with the adjustments mentioned in Section 7.1). The buyer with $t \in [p'/\bar{q}(p'), t_1]$ may consider to unilaterally announce $p < p'$. To see that there are sellers who offer to buy at this price, consider the following. For sellers with $q \leq p$, utility from selling at $p < p'$, i.e., $p - q$, is non-negative. Moreover, at price p', the probability of sale is strictly lower than one and changes only marginally if there is a buyer announcement. Therefore, sellers with $q \leq p$ will always offer to sell at $p < p'$. This is because the additional opportunity to sell at p weakly increases the expected utility. In case of offering to sell at p, the seller is able to sell her good at p with positive probability and she may still sell her good at p' with the given probability of sale. In case of not offering to sell at p, she may just sell her good at p' with the given probability of sale. Thus, as long as there are sellers with $q \leq p$, the buyer is able to buy at $p < p'$. In this case, utility from buying at $p < p'$ is given by $t\bar{q}(p) - p$. If the buyer unilaterally announces a price $p \geq p'$, all sellers with $q \leq p$ will offer their good at p. Therefore, the buyer's utility from unilaterally announcing $p \geq p'$ is given by $t\bar{q}(p) - p$. As a result, the buyer's utility from unilateral announcement p is given by $u(p; p', t)$ for $t \in [p'/\bar{q}(p'), t_1]$ as defined in Lemma 4. If p' maximizes $u(p; p', t)$, every buyer with type $t \in [p'/\bar{q}(p'), t_1]$ will unilaterally announce p' such that p' is robust to any buyer announcement.
Consider now the case where instead of a buyer announcement there is a seller announcement as defined in Definition 3 (with the adjustments mentioned in Section 7.1). Because the probability of sale is strictly lower than one at p', the seller with $q \leq p$ may consider to unilaterally announce a price $q \leq p < p'$. Moreover, the seller may also consider to unilaterally announce a price $p \geq p'$. However, the seller will not announce any price $p \neq p'$ if at these prices, no buyer is willing to offer to buy. A buyer's utility from buying at announced prices is given by $t\bar{q}(p) - p$ for $p \leq p'$ and $t\bar{q}(p') - p < t\bar{q}(p) - p$ for $p > p'$ and $t \in [t_0, t_1]$. Thus, if p' maximizes $u(p; p', t)$ for $t \in [t_0, t_1]$ as defined in Lemma 4, p' certainly maximizes the buyers' utility from offering to buy at the announced price p. If the condition in Lemma 4 is satisfied, no buyer is willing to buy at prices $p \neq p'$. In this case, the seller's utility from unilaterally announcing a price $p \neq p'$ is zero. She will therefore stick to p' such that p' is robust to any seller announcement.

To sum up, the condition in Lemma 4 is sufficient for p' with $D(p') < S(p')$ to be robust to any buyer and seller announcements. $\square$

It remains to determine the conditions under which Lemma 4 is satisfied. Let $p_t = \arg\max_p t\bar{q}(p) - p$ for buyer $t \in [t_0, t_1]$. Applying Athey (2002), p_t is the same for all t if utility is log-supermodular in (p, t). i.e.,

$$\frac{\partial}{\partial t} \frac{t\bar{q}(\hat{p}) - \hat{p}}{t\bar{q}(p) - p} = 0 \tag{17}$$

for $q_0 < p \leq q_1$ and $\hat{p} > p$. Rewriting (17) yields $\hat{p}/\bar{q}(\hat{p}) = p/\bar{q}(p)$. By replacing all the inequalities in the appendix with equalities, it can be easily seen that $\partial\varepsilon^S/\partial p = 0$ for $q_0 < p \leq q_1$ is a sufficient condition for a constant price-quality ratio. Under a constant price-quality ratio for $q_0 < p \leq q_1$, $t\bar{q}(p) - p = p(t/c - 1)$ for $q_0 < p \leq q_1$, where $c = p/\bar{q}(p)$. The first order condition reads $t/c - 1$ such that $t\bar{q}(p) - p$ is increasing for $t > c$ and constant for $t = c$ for prices $q_0 < p \leq q_1$. Thus, independently of p', $p_t = q_1$ for all $t > c$. If $p' = q_1$, no buyer with $t \in [c, t_1]$ has an incentive to unilaterally announce $p \neq p'$. Hence, given that $S(p') > D(p')$ and $\partial\varepsilon^S/\partial p = 0$ for $q_0 < p \leq q_1$, $p' = q_1$ is the unique refined equilibrium with excess supply. This proves Theorem 6.

The intuition underlying Theorem 6 is as follows. With a constant elasticity of supply, the inflow of high-quality sellers does not change with price. Even though the fraction of inflowing sellers decreases, the quality of these sellers increases. Hence, the increase in average quality caused by an increase in price is such that the price-quality ratio remains the same. Buyers therefore prefer higher over lower prices because higher prices imply a higher average quality at the same price-quality ratio. If, with excess supply at p', the buyer's utility of trading at p is everywhere increasing in price for $q_0 < p \leq q_1$ and decreasing for $p > q_1$, this buyer prefers to trade at $p = q_1$. Hence, no matter whether sellers or buyers unilaterally announce prices, the announced price will be $p = q_1$. As a result, p' is a refined equilibrium with excess supply as defined in Definition 6 if $p' = q_1$ with $S(p') > D(p')$.

For an illustration of Theorem 6, consider Figure IV.

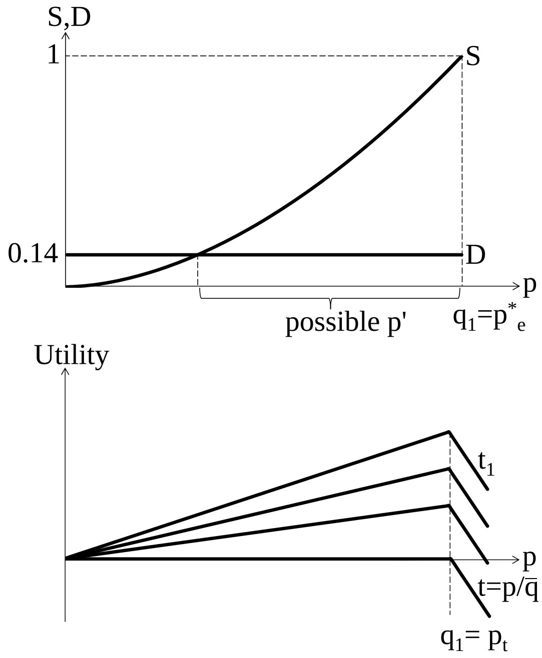

Figure IV: Example for a unique refined equilibrium with excess supply

In this example, quality $q \in [0,1]$ is distributed according to $f(q) = 1.8q^{0.8}$, whereas preference types $t \in [1.5, 1.9]$ are uniformly distributed. The upper part of Figure IV shows demand and supply and the possible range for p' for which $S(p') > D(p')$. The elasticity of supply is constant and equals 1.8. The lower part of Figure IV shows the utility of some buyers with type $t \in [p'/\bar{q}(p'), t_1]$ as a function of the unilaterally announced price p. Obviously, all buyers would unilaterally announce $p = q_1$ and sellers have no incentive to announce $p \neq q_1$. Hence, $p = q_1$ is the unique refined equilibrium with excess supply if this price is the result of the price-setting mechanism.

The unique refined equilibrium with excess supply requires a somewhat stronger condition on the elasticity of supply than the unique refined market-clearing equilibrium. This is because excess supply allows to unilaterally announce low prices, which is excluded in case of a market-clearing price.

The application of Theorem 3 reveals that, among standard distributions, the uniform and the Pareto distribution feature a constant elasticity of supply. Another example is the power distribution with $F(q) = q^a$, where a denotes the elasticity of supply.

Note that there may be other conditions under which $p_t = q_1$ for all $t \in [p'/\bar{q}(p'), t_1]$ and $S(p') > D(p')$ such that $p_e^* = q_1$. Consider the first order condition of the buyer's utility maximization problem given in Lemma 4, i.e.,

$$\frac{\partial}{\partial p}\left(t\bar{q}(p) - p\right) = t\varepsilon^S\left(1 - \frac{\bar{q}}{p}\right) - 1. \tag{18}$$

for $t \in [p'/\bar{q}(p'), t_1]$ and $p \geq q_0$. Clearly, the utility of these types is increasing for $q_0 < p \leq q_1$ and positive at $p = q_1$ if the elasticity of supply is everywhere high and average quality is everywhere relatively low. However, if in some price interval, the elasticity of supply is high, the average quality in this interval is also relatively high (as shown in Ewerhart and Feubli, 2012). Thus, it might be difficult to find a numerical example where indeed, the elasticity of supply is high and average quality is relatively low for $q_0 < p \leq q_1$.

We now briefly discuss equilibrium rationing. Based on the definition of credit rationing in Stiglitz and Weiss (1981), we define seller rationing in the adjusted set-up as follows. There is equilibrium seller rationing if among the sellers offering their good at p_e^*, some are able to sell their good and some are not, and some of the latter are not able to sell the good even if they offer a price $p < p_e^*$. Obviously, there may be seller rationing at $p_e^* = q_1$ with $S(p_e^*) > D(p_e^*)$.

8 Extension: Multidimensional seller types

So far, we have worked under the assumption that the quality of a seller's endowment perfectly describes her reservation price. In a richer model, the sellers' reservation prices may differ from endowment quality. Such a set-up applies, for example, to a situation where sellers not only differ in the quality of endowment, but also in their stock capacity. A seller with a small stock capacity may have a lower reservation value than a seller which offers the same quality, but has a higher stock capacity.[8]

In this section, we study the uniqueness of the market-clearing equilibrium in a set-up where seller types are two-dimensional.

Assume that sellers are characterized by a pair, consisting of the quality of their

[8] The used car wholesale market, studied by Genesove (1993), is an example for this set-up.

endowment q, and their reservation price v. Seller types are distributed according to some strictly positive and continuous density on $[v_0, v_1] \times [q_0, q_1]$, where $v_0 < v_1$ and $q_0 < q_1$. Let $f = F' > 0$ be the density of v. A seller is willing to offer her good for sale only if the market price p is weakly higher than her reservation value v. If p is the single market price, supply at this price is given by $S(p) = F(p)$. Moreover, average quality at this price is given by $\bar{q}(p) = E[q|v \leq p]$, whereas marginal quality is $m(p) = E[q \mid v = p]$. Similar to the case with one-dimensional seller types, demand at the single market price p is given by $D(p) = 1 - H(p/\bar{q}(p))$, where H is the fraction of buyers with preference type $\leq t$. In this set-up, the market-clearing equilibrium is as defined in Definition 1.

For at most one market-clearing equilibrium, supply and demand as functions of price have to cross at most once. Supply is zero for prices $p \leq v_0$, it is strictly increasing for prices $p \in [v_0, v_1]$ and constant and positive for prices $p > v_1$. Because average quality is not defined for $p < v_0$ and constant for $p \geq v_1$, demand is not defined for $p < v_0$ and strictly decreasing for prices $p \geq v_1$. Thus, for at most one market-clearing equilibrium, it suffices that demand is weakly declining for prices $p \in [v_0, v_1]$. This in turn requires $p/\bar{q}(p)$ to be weakly increasing in price for $p \in [q_0, q_1]$. Because a seller's quality of endowment and the reservation price may differ, a weakly increasing price-quality ratio requires not only a non-increasing elasticity of supply, but also an inelastic marginal quality. Denote the elasticity of marginal quality by $\varepsilon^m = \partial \log m(p)/\partial \log p$.

Theorem 7. Assume $\partial\varepsilon^S/\partial p \leq 0$ and $\varepsilon^m \leq 1$ for $p \in [v_0, v_1]$. Then, $\partial D/\partial p \leq 0$ for $p \geq v_0$, and the market-clearing equilibrium is unique.

Proof. The price-quality ratio $p/\bar{q}(p)$ is weakly increasing on the price interval $[v_0, v_1]$ if

$$\frac{\partial}{\partial p}\left[\frac{pF(p)}{\int_{q_0}^{p} m(q)f(q)dq}\right] \geq 0. \tag{19}$$

Rewriting (19) yields

$$\frac{\partial}{\partial p}\left[\frac{\int_{q_0}^{p} qf(q) + F(q)dq}{\int_{q_0}^{p} m(q)f(q)dq}\right] \geq 0. \tag{20}$$

Define $g(q, a) = (1-a)m(q)f(q)+a(qf(q)+F(q))$ for $a = 0, 1$. Then, inequality (20) amounts to the log-supermodularity of $\int_{q_0}^{p} g(q, a)dq$ in (p, a). Hence, $\partial/\partial p(p/\bar{q}) \geq$

0 can be reinterpreted as $\int_{q_0}^{p} g(q,a)dq$ satisfying the single crossing property. As log-supermodularity is stable under integration (cf., e.g., Athey, 2002), a sufficient condition for inequality (19) is

$$\frac{\partial}{\partial p}\left[\frac{pf(p)+F(p)}{m(p)f(p)}\right] \geq 0, \tag{21}$$

for all $p \in (v_0, v_1)$. Rewriting (21) by using the definitions of the elasticity of supply and the elasticity of marginal quality delivers

$$\frac{\partial \varepsilon^S/\partial p}{\varepsilon^S(1+\varepsilon^S)} \leq \frac{1}{p}(1-\varepsilon^m). \tag{22}$$

It clearly suffices to impose $\partial\varepsilon^S/\partial p \leq 0$ and $\varepsilon^m \leq 1$ for (22) to be satisfied and therefore, the price-quality ratio $p/\bar{q}(p)$ to be weakly increasing for $p \in [v_0, v_1]$. From the definition of demand $D(p)$ it follows that these two conditions are also sufficient to obtain a weakly downward-sloping demand and hence, a unique market-clearing equilibrium p^*.[9] $\square$

Intuitively, for demand to be decreasing in price, the marginal type with $t = p/\bar{q}(p)$ and therefore, the price-quality ratio has to increase with price for $p \in [v_0, v_1]$. Thus, as already seen in Section 4, this requires the increase in average quality to decrease with price. A non-increasing elasticity of supply and a low elasticity of marginal quality together lead to a decreasing inflow of sellers with high reservation prices, whose average quality $m(p)$ increases only slightly with price. Under these two conditions, the increase in average quality indeed decreases such that the price-quality ratio is increasing in price for $p \in [v_0, v_1]$.

9 Concluding remarks

In this paper, we offered simple and sufficient conditions for a unique market-clearing equilibrium in markets with asymmetric information. We showed that the market-

[9] With two-dimensional seller types, it suffices to impose $\varepsilon^S < 1/t_1$ and $m(p) \leq p$ for $p \geq p^*$ for a unique refined market-clearing equilibrium. To prove this result, one rewrites $\partial\bar{q}/\partial p < 1/t_1$ (see Section 6) using the definitions of elasticity of supply and marginal quality. For a unique refined equilibrium $p_e^* = q_1$ with $S(p_e^*) > D(p_e^*)$, it suffices to add the condition $\varepsilon^m = 1$ in Theorem 6.

clearing price is unique if the elasticity of supply is non-increasing. Our study revealed that many quality distributions feature a unique market-clearing equilibrium. In case sellers possess heterogeneous preferences, it suffices to additionally assume an elasticity of marginal quality lower than one. To refine the market-clearing concept, we introduced the possibility that either a single seller or a single buyer can decide whether to trade at the unique market-clearing price. It turned out that in this case, the unique market-clearing price is robust to unilateral price announcements if the elasticity of supply is small. We extended our analysis by introducing the possibility that units are traded at some price which is not necessarily the market-clearing price. Thereby, we were seeking conditions to prevent sellers and buyers to unilaterally announce a price different from the suggested market price such that this market price is a refined equilibrium. We found that in this set-up, a constant elasticity of supply is sufficient for a unique refined equilibrium with excess supply. Our analysis shows that there is no refined equilibrium with excess demand.

Appendix

In this appendix, we apply Van den Berg (1994) to show that $\partial\varepsilon^S/\partial p \leq 0$ indeed suffices for $\mathrm{d}\log\bar{q}(p)/\mathrm{d}\log p \leq 1$ to be satisfied for $p \in [q_0, q_1]$.
By using the fact that $\mathrm{d}\log\bar{q}(p) = \mathrm{d}\bar{q}(p)/\bar{q}(p)$ and $\mathrm{d}\log p = \mathrm{d}p/p$, we write $1 \geq \mathrm{d}\log\bar{q}(p)/\mathrm{d}\log p$ as $\bar{q}(p)/p \geq \mathrm{d}\bar{q}(p)/\mathrm{d}p$ for all $p \in [q_0, q_1]$. Rewriting it yields

$$\frac{\bar{q}(p)}{p} \geq \frac{f(q)}{F(p)}\left(p - \bar{q}(p)\right). \tag{A.1}$$

By multiplying both sides with $pF(p)$, one gets $F(p)\bar{q}(p) \geq pf(p)(p-\bar{q}(p))$ for $p \in [q_0, q_1]$, which in turn can be rewritten as

$$\int_{q_0}^{p} qf(q)dq \geq pf(p)\left(p - \bar{q}(p)\right) \tag{A.2}$$

for $p \in [q_0, q_1]$. Expanding the term in brackets on the right-hand side of (A.2) by $F(p)/F(p)$ and then rewriting the inequality by using the fact that $F(p)(p - \bar{q}(p)) = \int_{q_0}^{p}(p-q)dF(q) = \int_{q_0}^{p} F(q)dq$ yields

$$\int_{q_0}^{p} qf(q)dq \geq \int_{q_0}^{p} \frac{pf(p)}{F(p)}F(q)dq \tag{A.3}$$

for $p \in [q_0, q_1]$. The rearrangement of terms results in

$$\int_{q_0}^{p}\left[\frac{qf(q)}{F(q)} - \frac{pf(p)}{F(p)}\right]F(q)dq \geq 0 \tag{A.4}$$

for $p \in [q_0, q_1]$. At $q = p$, the term in brackets of inequality (A.4) equals zero. Hence, if $qf(q)/F(q) \geq pf(p)/F(p)$ for all $q < p$, inequality (A.4) holds. A sufficient condition for $qf(q)/F(q) \geq pf(p)/F(p)$ for all $q < p$ to hold is $qf(q)/F(q)$ to be decreasing in q, which is nothing else but the requirement that $\partial\varepsilon^S/\partial p \leq 0$ for $p \in [q_0, q_1]$.

References

Adriani, F., and L. Deidda (2009): "Price Signaling and the Strategic Benefits of Price Rigidities," *Games and Economic Behavior*, 67, 335–350.

Akerlof, G. (1970): "The Market for "Lemons": Quality Uncertainty and the Market Mechanism," *The Quarterly Journal of Economics*, 84, 488–500.

Akerlof, G., and J. Yellen (1986): "Introduction," in *Efficiency Wage Models of the Labor Market*, ed. by G. Akerlof, and J. Yellen, pp. 1–21. Cambridge University Press Cambridge.

Arnold, L., and J. Riley (2009): "On the Possibility of Credit Rationing in the Stiglitz-Weiss Model," *American Economic Review*, 99, 2012–2021.

Athey, S. (2002): "Monotone Comparative Statics Under Uncertainty," *The Quarterly Journal of Economics*, 117, 187–223.

Bagnoli, M., and T. Bergström (2005): "Log-Concave Probability and its Applications," *Economic Theory*, 26, 445–469.

Bester, H. (1988): "Qualitative Uncertainty in a Market with Bilateral Trading," *Scandinavian Journal of Economics*, 90, 415–434.

Bester, H., and K. Ritzberger (2001): "Strategic Pricing, Signalling, and Costly Information Acquisition," *International Journal of Industrial Organization*, 19, 1347–1361.

Bigelow, J. P. (1990): "Efficiency and Adverse Selection," *Journal of Economic Theory*, 52, 380–405.

Blouin, M. R. (2003): "Equilibrium in a Decentralized Market with Adverse Selection," *Economic Theory*, 22, 245–262.

Ellingsen, T. (1997): "Price Signals Quality: The Case of Perfectly Inelastic Demand," *International Journal of Industrial Organization*, 16, 43–61.

Ewerhart, C. (2011): "Optimal Design and ρ-Concavity," *Economic Theory*, p. forthcoming.

Ewerhart, C., and P. Feubli (2012): "Monotone Comparative Statics in Markets with Asymmetric Information," *Working paper.*

Genesove, D. (1993): "Adverse Selection in the Wholesale Used Car Market," *Journal of Political Economy*, 101, 644–665.

Guerrieri, V., R. Shimer, and R. Wright (2010): "Adverse Selection in Competitive Search Equilibrium," *Econometrica*, 78, 1823–1862.

Inderst, R., and H. Müller (2002): "Competitive Search Markets for Durable Goods," *Economic Theory*, 19, 599–622.

King, T. (2008): "Discipline and Liquiditiy in the Interbank Market," *Journal of Money, Credit and Banking*, 40, 295–317.

Ponsatí, C., and J. Sákovics (2008): "Queues, not just Mediocrity: Inefficiency in Decentralized Markets with Vertical Differentation," *International Journal of Industrial Organization*, 26, 998–1014.

Rose, C. (1993): "Equilibrium and Adverse Selection," *RAND Journal of Economics*, 24, 559–569.

Stiglitz, J. (1976): "Prices and Queues as Screening Devices in Competitive Markets," Discussion Paper 212, Institute for Mathematical Studies in the Social Sciences Stanford University.

Stiglitz, J., and A. Weiss (1981): "Credit Rationing in Markets with Imperfect Information," *The American Economic Review*, 71, 393–410.

Van den Berg, G. (1994): "The Effects of Changes of the Job Offer Arrival Rate on the Duration of Unemployment," *Journal of Labor Economics*, 12, 478–498.

Wilson, C. (1979): "Equilibrium and Adverse Selection," *The American Economic Review*, 69, 313–317.

——— (1980): "The Nature of Equilibrium in Markets with Adverse Selection," *The Bell Journal of Economics*, 11, 108–130.

Chapter 2

Monotone Comparative Statics in Markets with Asymmetric Information

joint with Christian Ewerhart

1 Introduction

In a market where the units of a good differ in quality, sellers may be better informed about the quality of a unit than buyers. Typically, sellers know the quality of their endowment, whereas buyers only observe average quality of units offered for sale. Because buyers cannot distinguish units of different quality, all units sell at the same price. Assume that the quality of a unit equals the seller's reservation price. Thus, a seller offers a unit for sale only if the market price is weakly higher than quality. As a result, the market with a single price is characterized by adverse selection, i.e., a price only attracts units of relatively low quality. This in turn may cause the market outcome to be inefficient in terms of what government, investors, or market participants define as efficiency. For example, one inefficiency may be that units of high quality are not sold, whereas units of low quality sell at a relatively high price. To improve the market outcome, mechanisms such as screening (e.g., Rothschild and Stiglitz, 1976), signaling (e.g., Spence, 1976), licensing (e.g., Leland, 1979), or other forms of governmental interventions may be introduced. However, the optimal design of such mechanisms requires the understanding of how a change in the underlying market conditions affects the market outcome. Here, comparative statics analysis of equilibrium trade volume and price is essential.

In this paper, we offer simple and sufficient conditions for monotone comparative statics of the unique market-clearing equilibrium in the standard set-up of markets with asymmetric information.[1] Our first main result addresses equilibrium trade volume. Intuitively, for equilibrium trade volume to unambiguously decrease, it is sufficient that exogenous changes in the type distributions of market participants have a negative effect on both supply and demand. We show that if exogenous parameter changes lower supply, the elasticity of supply, as well as the aggregate willingness to buy, equilibrium trade volume decreases. However, these three conditions together do not allow clear-cut predictions of changes in the equilibrium price. Our second main result therefore addresses equilibrium price. Intuitively, for equilibrium price to increase, it suffices that the exogenous parameter changes have a positive effect on demand and a negative effect on supply. We show that this translates into the requirement that the exogenous changes lower supply, and increase the elasticity of supply and the aggregate willingness to buy. The combination of these conditions leads to an unambiguous increase in the equilibrium price. However, it does not allow clear-cut predictions of changes in equilibrium trade volume.[2]

We also study comparative statics of equilibrium trade volume and excess supply in a set-up where market-clearing as an equilibrium condition is dropped. Equilibrium excess supply is defined as the positive difference between equilibrium supply and demand. Equilibrium trade volume is therefore determined by equilibrium demand. In our third main result we show that in this set-up, equilibrium trade volume decreases if the reaction of the elasticity of supply and aggregate willingness to buy to the exogenous change is negative such that there is a decline in demand. In our fourth main result we show that if the exogenous change is such that it lowers demand and the reaction of supply is non-negative, equilibrium excess supply increases.

The rest of the paper is organized as follows. Section 2 discusses related literature. The model is outlined in Section 3. In Sections 4 and 5, we derive sufficient conditions for monotone comparative statics of trade volume and price. In Section 6

[1] For the standard set-up, see Akerlof (1970) and Wilson (1980).

[2] These results have appeared in a preliminary form in our IEW Working Paper No. 455.

we study the comparative statics of excess supply. Section 7 discusses empirical evidence. Section 8 concludes.

2 Related Literature

Our study relates to several other contributions to comparative statics analysis in markets with asymmetric information or under uncertainty.

In an applied study, Van den Berg (1994) examines the effect of changes in the job arrival rate on the exit rate out of unemployment in job search models. In his set-up, the probability of accepting a job offer depends on the job arrival rate and the agent's reservation wage, which itself is a function of the job arrival rate. Van den Berg (1994) presents conditions under which the conditional probability of accepting an offer is non-decreasing in the job arrival rate. Besides using a different set-up, Van den Berg's (1994) study differs from ours insofar as Van den Berg considers conditions for monotone comparative statics of the individual participation decision (probability of accepting a job offer), whereas we are also interested in the market outcome. Moreover, in our set-up, individual reservation price and maximum willingness to pay are exogenous.

Greenwald (1986) considers comparative statics in a two-period labor market model. Employers are assumed to learn the employees' individual ability during the first period and they are given the possibility to adjust the individual wage for period two. Depending on the wage offers for period two, workers decide whether to stay with the initial employer or to change job by entering the secondhand job market. Greenwald (1986) shows that in this set-up, an increase in the fraction of high-ability workers in the population of job applicants both increases equilibrium wage and job turnover in the secondhand labor market. Moreover, a mean-preserving spread in the ability distribution lowers equilibrium wage in this market. Greenwald's (1986) set-up goes beyond the standard set-up of markets with asymmetric information. This is because he uses a dynamic framework and assumes that workers with reservation wages higher than the secondhand market wage will enter the secondhand market with positive probability. In addition, an employer's demand in the secondhand market not only depends on average ability in this market, but also on the mix of abilities in her individual labor force.

Comparative statics properties of the market outcome have also been studied with respect to the information structure. Kessler (2001) shows that market performance (aggregate surplus) is non-monotonic in the number of uninformed sellers. In Levin (2001), sellers observe a private signal for quality, which is positively correlated to the true quality. Levin (2001) concludes that trade is non-monotonic in the precision of signals. Daley and Green (2010) consider a dynamic set-up where buyers periodically receive information on the value of assets offered for sale. Depending on buyers' beliefs about quality, more precise information either leads to less or more delay in trade. In Kurlat (2010), buyers learn about quality by observing past transactions. Less learning, caused by a market downturn, worsens future adverse selection and may lead to a market breakdown. These studies differ from our study insofar as we focus on changes in the market participants' type distributions rather than on changes in the asymmetry of information.

From a technical point of view, our paper partially builds on Athey (2002). Athey (2002) considers stochastic optimization problems and establishes necessary and sufficient conditions for comparative statics predictions. In particular, Athey (2002) studies the choice vector of agents who maximize expected utility. Thereby, expected utility averages over state-contingent utilities, where the distribution of states depends on an individual exogenous parameter. Athey (2002) presents necessary and sufficient conditions under which the agent's choice vector is non-decreasing in the individual exogenous parameter.

There exists also empirical literature on comparative statics in markets with asymmetric information. We will discuss some of these contributions in Section 7.

3 Set-up

In this section, we present the set-up and discuss the unique market-clearing equilibrium and the unique refined equilibrium. We thereby follow Ewerhart and Feubli (2012).

3.1 Market-clearing: price-taking agents

There is a market for an indivisible good and both buyers and sellers are price-takers. We will later relax the price-taking assumption.

In this market, there is a continuum of sellers, whose population size equals N. Each seller possesses one unit of the good and wants to sell it in the market. Endowments across sellers differ in quality q, where $q \in [q_0, q_1]$ with $q_0 < q_1$. Quality q is distributed according to some density f which is strictly positive on $[q_0, q_1]$. We assume that q not only describes the quality of the seller's endowment, but also perfectly describes the seller's reservation price. Thus, a seller is willing to sell her endowment only if the offered price p is weakly higher than quality q. From this it follows that if there is a single market price p, supply at this price is given by $S(p) = N \cdot F(p)$, where F denotes the cumulative distribution function of q.

There is also a continuum of buyers. The size of the buyer population is given by J. Buyers have no endowment and want to buy one unit of the good. In contrast to sellers, buyers cannot observe individual quality of some unit. However, they are able to infer average quality of units on offer from price p. It is assumed that buyers differ in their preference for quality, i.e., each buyer has a privately observed individual preference type $t \in [t_0, t_1]$ with $t_0 < t_1$ and $t_1 > 1$. Preference types are distributed according to some density h. A buyer is willing to buy a unit if $t\bar{q}(p) \geq p$, where $\bar{q}(p) = E[q \mid q \leq p]$ is the average quality of units on offer. Let $\bar{q}(p) = q_0$ at price $p = q_0$. Define $H(t)$ as the fraction of buyers with preference type $\leq t$. If t is the lowest buyer type which is willing to buy at some price p, the aggregate willingness to buy at this price is described by $J(1 - H(t))$, i.e., the mass of buyers with a preference type above t. Consequently, if p is the single market price, demand at this price is given by $D(p) = J(1 - H(p/\bar{q}(p)))$.

Given that both buyers and sellers are price-takers, we define the market-clearing equilibrium as follows (see also Ewerhart and Feubli, 2012, for example).

Definition 1. A market-clearing equilibrium is a price p^* for which $S(p^*) = D(p^*)$, i.e.,

$$N \cdot F(p^*) = J(1 - H(p^*/\bar{q})). \tag{1}$$

In equilibrium, buyers and sellers willing to trade at p^* are randomly matched and units are bilaterally traded at the market-clearing price.

We now present conditions for the uniqueness of the market-clearing equilibrium, thereby referring to Ewerhart and Feubli (2012).

3.2 Uniqueness of the market-clearing equilibrium

For the market-clearing equilibrium to be unique, demand and supply as functions of price are required to intersect at most once. From the definition of supply follows that it is zero for prices $p \leq q_0$, strictly increasing for prices $p \in [q_0, q_1]$, and constant and positive for prices $p > q_1$. Demand depends on the price-quality ratio $p/\bar{q}(p)$. Because average quality is not defined for $p < q_0$ and constant for prices $p > q_1$, demand is not defined for prices below q_0, positive at price $p = q_0$, and strictly decreasing for prices above q_1. Hence, as already shown by Ewerhart and Feubli (2012), for a unique market-clearing equilibrium it suffices that demand is weakly declining for prices $p \in [q_0, q_1]$. This in turn requires the price-quality ratio to increase in price for $p \in [q_0, q_1]$. Define the elasticity of supply as $\varepsilon^S = (\partial S(p)/\partial p) \cdot (p/S(p))$. For an increasing price-quality ratio, it is sufficient to impose a non-increasing elasticity of supply (see Theorem 1 in Ewerhart and Feubli, 2012).

Lemma 1. Assume $\partial \varepsilon^S/\partial p \leq 0$ for prices $p \in [q_0, q_1]$. Then $\partial D/\partial p \leq 0$ for $p \geq q_0$, and the market-clearing equilibrium is unique.

The proof of Lemma 1 is equal to the proof of Theorem 1 in Ewerhart and Feubli (2012) and is therefore omitted.
Intuitively, to get a declining demand function, the marginal buyer with $t = p/\bar{q}(p)$ is required to be increasing in price. For the price-quality ratio $p/\bar{q}(p)$ to increase with price, it suffices that the increase in average quality decreases with price. This in turn requires that the inflow of sellers with high quality goods decreases with price. A weakly decreasing elasticity of supply reflects exactly this requirement.

4 Comparative statics: Volume

In this section, we conduct a comparative statics analysis of trade volume in the unique market-clearing equilibrium. In particular, we derive sufficient conditions on the exogenous changes in the type distributions of market participants for equilibrium trade volume to unambiguously decline.

For expository reasons, we introduce two independent markets M_A and M_B, each as described in Section 3.1. It is assumed that the intervals $[q_0, q_1]$ and $[t_0, t_1]$ are common to both markets. We capture all exogenous changes by assuming that market M_A is characterized by the initial vector $(N_A, J_A, F_A(.), H_A(.))$, whereas market M_B is characterized by the vector $(N_B, J_B, F_B(.), H_B(.))$, which may differ from the initial vector in market M_A. The elasticity of supply is non-increasing in both markets so that they feature a unique market-clearing equilibrium (see Lemma 1).
Before we turn to the analysis of equilibrium trade volume, we discuss an important feature of comparative statics in markets with asymmetric information. As will be shown below, average quality decreases for prices $p > q_0$, if for these prices, the elasticity of supply decreases.

Lemma 2. Assume $\varepsilon_B^S(p) \leq \varepsilon_A^S(p)$ for all $p > q_0$. Then $\bar{q}_B(p) \leq \bar{q}_A(p)$ for all $p > q_0$.

Because the proof of Lemma 2 is somewhat lengthy, it is left to the appendix. For intuitive insights, consider the following. A marginal increase in price causes average quality to increase. This is because the marginal price increase leads to an inflow of sellers with relatively high quality. If the elasticity of supply at price p is smaller in market M_B than in market M_A, a marginal increase in price leads to a smaller inflow of sellers in market M_B than in market M_A. Thus, the increase in average quality in market M_B is smaller than in market M_A. Given that $\varepsilon_B(p) \leq \varepsilon_A(p)$ for all prices $p > q_0$, average quality is weakly lower in market M_B than in market M_A for prices $p > q_0$.
We now turn to the comparative statics analysis of equilibrium trade volume.

Theorem 1. Assume that

1) $N_B F_B(p) \leq N_A F_A(p)$ for all p,

2) $f_B(p)/F_B(p) \leq f_A(p)/F_A(p)$ for all $p > q_0$,

3) $J_B\,(1 - H_B(t)) \leq J_A\,(1 - H_A(t))$ for all t.

Then, $V_B^* \leq V_A^*$, where V_i^* denotes the respective trade volume in the unique market-clearing equilibrium in market M_i with $i = A, B$.

Proof. Let p_A^* and p_B^* denote the respective equilibrium prices in markets M_A, M_B. If $p_B^* \leq p_A^*$, then $V_B^* = S_B(p_B^*) \leq S_A(p_B^*) \leq S_A(p_A^*) = V_A^*$ from condition 1. In case

$p_B^* > p_A^*$, $D_A(p_B^*) \leq D_A(p_A^*) = V_A^*$ from Lemma 1 (proof can be found in Ewerhart and Feubli, 2012). It remains to show that $V_B^* = D_B(p_B^*) \leq D_A(p_B^*)$, which, by condition 3, reduces to the requirement that $\bar{q}_B(p) \leq \bar{q}_A(p)$ for any $p > q_0$. By Lemma 2, condition 2 is sufficient for $\bar{q}_B(p) \leq \bar{q}_A(p)$ to be satisfied for $p > q_0$. Hence, conditions 1 to 3 are sufficient for $V_B^* \leq V_A^*$. □

Intuitively, for equilibrium trade volume to unambiguously weakly decrease as one moves from market M_A to market M_B, the exogenous parameter changes are required to have a non-positive effect on demand and supply. Clearly, a decline in both seller and buyer population sizes has, ceteris paribus, a non-positive effect on supply and demand. The supply reaction is also non-positive if (in addition) the quality distribution changes in a way that it first-order stochastically dominates the initial one (condition 1). In this case, the demand reaction is non-positive if two conditions are satisfied. Demand depends positively on both aggregate willingness to buy and average quality. Thus, if (in addition to the exogenous change in buyer population size), there is a weak decline in both the aggregate willingness to buy (condition 3) and the elasticity of supply (condition 2), the demand reaction is non-positive. It follows that the conditions in Theorem 1 are sufficient for equilibrium trade volume to be unambiguously weakly lower in market M_B than in market M_A. Note that for $V_B^* < V_A^*$, it is sufficient that one condition in Theorem 1 is a strict inequality. The upper part of Figure I illustrates Theorem 1.

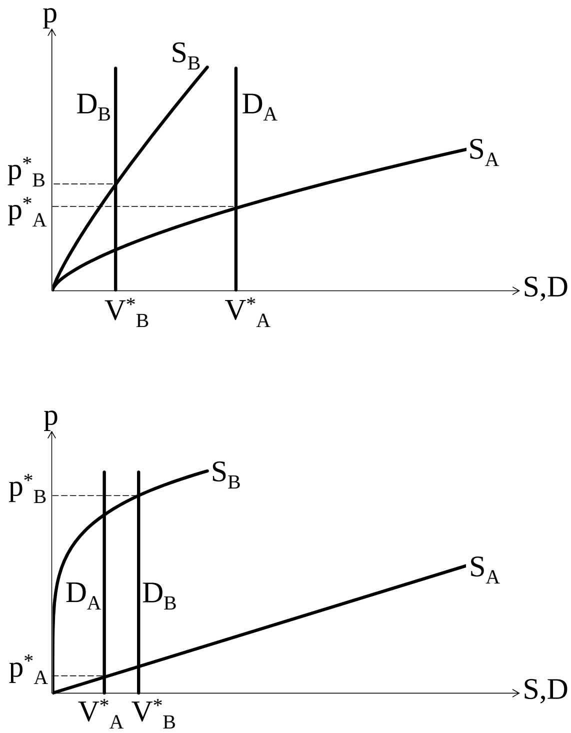

Figure I: Comparative statics of equilibrium trade volume

In Figure I, both markets M_A and M_B feature a power distribution for quality with $f_i(q) = \varepsilon_i q^{\varepsilon_i - 1}$ for $i = A, B$ and $q \in [0, 1]$, and a common uniform distribution for buyer preferences. By calculation one can check that the price-quality ratio $p/\bar{q}(p)$ in market M_i is constant and equal to $1+1/\varepsilon_i$ for $i = A, B$. Thus, demand in market M_i is perfectly inelastic. In the upper part of Figure I, parameter values are chosen such that the conditions in Theorem 1 are all satisfied with strict inequality.[3] On the supply side, F_B first-order stochastically dominates F_A and supply is lower in M_B than in M_A. The lower aggregate willingness to buy and the lower elasticity of supply imply a lower demand in market M_B than in market M_A. As a result,

[3]In particular, $(N_A, J_A, F_A, H_A) = (80, 40, q^{1.5}, (t-1)/1.2)$ and $(N_B, J_B, F_B, H_B) = (15, 20, q^{1.2}, (t-1)/1.2)$, where $[q_0, q_1] = [0, 1]$ and $[t_0, t_1] = [1, 2.2]$.

equilibrium trade volume in market M_B is strictly lower than in market M_A. The lower part of Figure I shows the case where condition 2 in Theorem 1 is violated, whereas conditions 1 and 3 are satisfied with strict inequality. In particular, parameter values are chosen such that the elasticity of supply in market M_B is higher than in market M_A. The lower part of Figure I shows an example where this leads to a higher equilibrium trade volume in market M_B than in market M_A. Generally, the violation of condition 2 in Theorem 1 leads to ambiguous comparative statics results for trade volume. Because average quality is higher and the aggregate willingness to buy is lower in market M_B than in market M_A, demand D_B may be either higher or lower than demand D_A. Thus, clear-cut results for equilibrium trade volume are no longer feasible. Similar arguments for conditions 1 and 3 show that unambiguous predictions for equilibrium trade volume require all three conditions in Theorem 1 to be simultaneously satisfied.

There is an additional possibility for equilibrium trade volume to decline. If the exogenous changes have a strongly negative effect on demand and a small but positive effect on supply or vice versa, equilibrium trade volume may still decrease. In this case, the conditions on the exogenous parameter changes are required to determine the sign and *size* of the effects on supply and demand. However, the size of the effect on supply and demand depends on the specific parameter values and ultimately remains a numerical question. Therefore, this possibility will not be discussed in this paper.

We now turn to the comparative statics analysis of the equilibrium price.

5 Comparative statics: Price

Consider markets M_A and M_B as introduced in Section 4.

Theorem 2. Assume that

4) $N_B F_B(p) \leq N_A F_A(p)$ for all p,

5) $f_B(p)/F_B(p) \geq f_A(p)/F_A(p)$ for all $p > q_0$

6) $J_B\,(1 - H_B(t)) \geq J_A\,(1 - H_A(t))$ for all t.

Then, $p_B^* \geq p_A^*$, where p_i^* denotes the respective price in the unique market-clearing equilibrium in market $M_i = M_A, M_B$.

Proof. To provoke a contradiction, assume $p_B^* < p_A^*$. Then, from condition 4 follows that $S_B(p_B^*) \leq S_A(p_B^*) \leq S_A(p_A^*)$. On the other hand, $D_B(p_B^*) \geq D_A(p_B^*) \geq D_A(p_A^*)$. The second inequality follows from Lemma 1 (for proof see Ewerhart and Feubli, 2012). The first inequality follows from conditions 5 and 6. Indeed, because of condition 6, it suffices to show that $p/\bar{q}_A(p) \geq p/\bar{q}_B(p)$ for $p = p_B^*$. By Lemma 2, condition 5 is sufficient for this inequality to be satisfied. Because $D_A(p_A^*) = S_A(p_A^*) = V_A^*$ and $D_B(p_B^*) = S_B(p_B^*) = V_B^*$, $D_A(p_A^*) \leq D_A(p_B^*) \leq D_B(p_B^*) = S_B(p_B^*) \leq S_A(p_B^*) \leq S_A(p_A^*)$. But then, necessarily, *all* the weak inequalities must be equalities. In particular, $S_A(p_A^*) = S_A(p_B^*) = D_A(p_B^*) = D_A(p_A^*)$. From the uniqueness of the market-clearing price follows that $p_A^* = p_B^*$, which is the desired contradiction. Hence, conditions 4 to 6 are sufficient for $p_B^* \geq p_A^*$. $\square$

Intuitively, for equilibrium price to unambiguously weakly increase as one moves from market M_A to market M_B, it suffices that the exogenous changes in the market participants' type distributions have a non-positive effect on supply and a non-negative effect on demand. On the one hand, a decline in the size of the seller population and an increase in the size of the buyer population, ceteris paribus, suffice for supply to decrease and demand to increase. On the other hand, (additional) changes in quality and preference distributions may also have the desired effects on supply and demand. As we have already seen in Section 4, the supply reaction is non-positive if (in addition) the quality distribution changes such that it first-order stochastically dominates the initial one (condition 4). The demand reaction is non-negative if (in addition to the exogenous change in buyer population size), the aggregate willingness to buy and the elasticity of supply weakly increase (condition 6 and 5, respectively). It follows that the conditions in Theorem 2 are sufficient for equilibrium price to be unambiguously weakly higher in market M_B than in market M_A. Again, one strict inequality in Theorem 2 suffices for $p_B^* > p_A^*$. The upper part of Figure II illustrates Theorem 2.

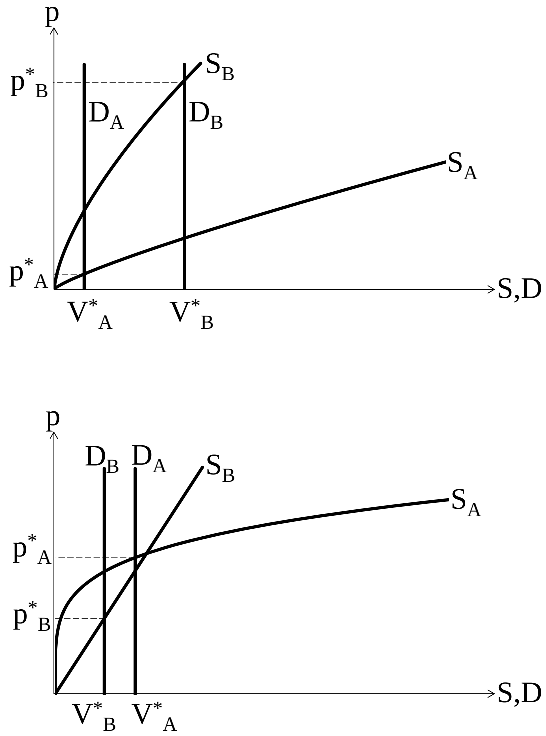

Figure II: Comparative statics of equilibrium price

In Figure II, quality and buyer preferences in markets M_A and M_B follow a power and a uniform distribution, respectively. Consider first the upper part of Figure II. Even though supply S_B is lower than supply S_A, the distribution F_B is chosen such that the elasticity of supply and therefore, average quality is higher in market M_B than in M_A. In combination with a higher aggregate willingness to buy in market M_B, demand D_B exceeds D_A for all prices $q_0 < p \leq q_1$.[4] As a result, the unique equilibrium price is higher in M_B than in M_A. The lower part of Figure II illustrates the case where conditions 4 and 5 in Theorem 2 are violated. More precisely, supply S_B exceeds supply S_A at low prices and the elasticity of supply is much lower in M_B than in M_A. However, the aggregate willingness to buy is higher in market M_B than in market M_A (condition 6). Whether demand D_B exceeds D_A in this case

[4] In particular, $(N_A, J_A, F_A, H_A) = (80, 10, q^{1.2}, (t-1)/1.2)$ and $(N_B, J_B, F_B, H_B) = (15, 30, q^{1.5}, (t-1)/1.2)$, where $[q_0, q_1] = [0, 1]$ and $[t_0, t_1] = [1, 2.2]$.

ultimately depends on the parameter values. The lower part of Figure II presents an example where $D_B < D_A$ for $q_0 < p \leq q_1$ and $p^*_B < p^*_A$. In general, as long as some of the conditions in Theorem 2 are violated, comparative statics of the equilibrium price remain ambiguous.

The comparative statics analysis of equilibrium price reveals that the conditions in Theorem 1 only allow to unambiguously predict differences in equilibrium trade volume, but leave predictions of the equilibrium price ambiguous. For illustration, consider Figure III.

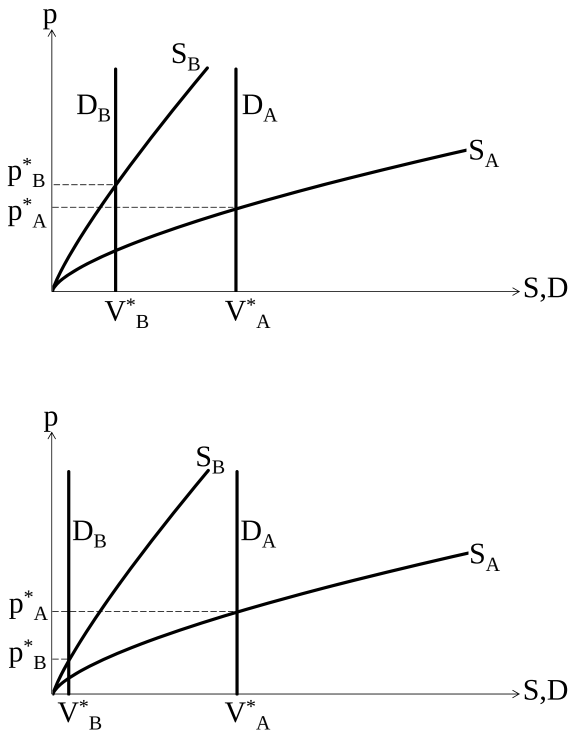

Figure III: The conditions in Theorem 1 do not unambiguously predict equilibrium price

The upper part of Figure III shows the same example for Theorem 1 as the upper part of Figure I. The example in the lower part of Figure III differs from the example in the upper part only insofar as the value of J_B is lower.[5] Hence, the example in the lower part still satisfies all conditions in Theorem 1 such that $V_B^* < V_A^*$. However, in contrast to the example in the upper part, $p_B^* < p_A^*$ in the lower part. Similarly, the conditions in Theorem 2 only allow to predict differences in equilibrium prices, but not differences in equilibrium trade volumes. To illustrate this finding, we present a example which slightly differs from the examples used so far. Consider Figure IV.

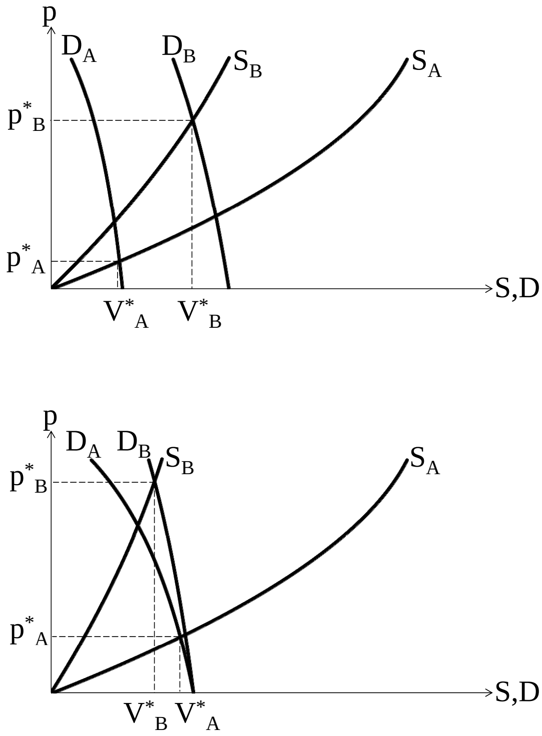

Figure IV: The conditions in Theorem 2 do not unambiguously predict equilibrium volume

The upper part in Figure IV shows an example which satisfies all the conditions in Theorem 2 such that $p_B^* > p_A^*$.[6] Moreover, parameter values are chosen such that $V_B^* > V_A^*$. The example in the lower part of Figure IV differs from the example in

[5]In particular, $J_B = 20$ in the upper part and $J_B = 5$ in the lower part.

[6]In the upper part of Figure IV, $(N_A, J_A, F_A, H_A) = (80, 20, 2/3(2.5q - q^2), t - 1.8)$ and $(N_B, J_B, F_B, H_B) = (40, 50, 4/3(q - q^2/4), t - 1.8)$, where $q \in [0, 1]$ and $t \in [1.8, 2.8]$.

the upper part only insofar as J_A is higher, and J_B as well as N_B are lower in the lower part.[7] Hence, the example in the lower part still satisfies all the conditions in Theorem 2 such that $p_B^* > p_A^*$. However, in the lower part, $V_B^* < V_A^*$.

6 Comparative statics: Excess supply

The underlying intuition of the market-clearing equilibrium concept is that competition among sellers and buyers drives the price towards the market-clearing level.[8] However, competition among market participants may not always be perfect such that market-clearing may not occur. This could be the case if, for example, average quality as a function of price is such that some buyers with price-setting power may prefer to buy at a price higher than the market-clearing price.[9] To be able to study comparative statics of equilibrium excess demand or supply, we adjust the set-up used so far and relax the market-clearing as well as the price-taking assumption. We thereby stick to Ewerhart and Feubli (2012).

6.1 Non-clearing: a single agent with price-setting power

Now, the unique price we are interested in does not result from a market-clearing process. Instead, we assume that goods are supposed to be traded at a unique price p' which is the result of some price-setting mechanism. At this price, there may be either excess demand where there are more buyers than sellers willing to trade, i.e., $D(p') > S(p')$. Or there may be excess supply where there are more sellers than buyers willing to trade, i.e., $D(p') < S(p')$. Moreover, we now allow for the possibility that one agent is given price-setting power, while all the other agents are still price-takers.

Suppose that a single buyer is given some price-setting power. In this case, the market works as follows.

Definition 2 (Buyer announcement). Assume that there is a price p' for which $D(p') \neq S(p')$. Before buyers are matched with sellers and any trade takes place,

[7] In particular, $(J_A, N_B, J_B) = (40, 25, 40)$ in the lower part of Figure IV.

[8] For a thorough discussion of market-clearing and price-taking behavior, see Mas-Colell, Whinston, and Green (1995).

[9] Sellers always prefer to sell at higher prices.

one single buyer who is willing to buy at p', i.e., a buyer with $t \in [p'/\bar{q}(p'), t_1]$, is chosen at random. This buyer has the possibility to announce the price at which she prefers to buy. The announced price may differ from price p'. However, if the buyer is indifferent between p' and any other price p, the buyer sticks to p'. We assume that all sellers can costlessly compare the announced price p with p'. Moreover, we assume that all sellers who prefer p over p' are equally likely to sell their unit at p. Those sellers which cannot sell at the announced price may sell their endowment at p'.

Suppose now that instead of a single buyer a single seller is given some price-setting power. In this case, the market works as follows.

Definition 3 (Seller announcement). Assume that there is a price p' for which $D(p') \neq S(p')$. Before sellers and buyers are matched and units are traded, a single seller willing to sell at p', i.e., a seller with $q \leq p'$, is randomly chosen. This seller has the possibility to announce the price at which she prefers to sell. The announced price may differ from p'. However, if the seller is indifferent between p' and any other price, she will announce p'. We assume that all buyers can costlessly compare the announced price p with p'. Moreover, we assume that buyers expect the quality of the single seller announcing p to be $\bar{q}(p)$ for $p \leq p'$ and $\bar{q}(p')$ for $p > p'$. All buyers who prefer p over p' are equally likely to buy the unit at p. Those buyers who cannot buy at p may buy a unit at p'.

We exclude the discussion of a refined equilibrium with excess demand because, as Ewerhart and Feubli (2012) show, such an equilibrium does not exist. We define the refined equilibrium with excess supply as follows.

Definition 4. A price p' is a refined equilibrium price p_e^* with excess supply if

1) $D(p') < S(p')$ and

2) p' is robust to any unilateral announcement of buyers with $t \in [p'/\bar{q}(p'), t_1]$ and sellers with $q \leq p'$, i.e.,

$$p' = \arg\max_p u(p; p', t), \tag{2}$$

where $u(p; p', t) = t\bar{q}(p) - p$ for $t \in [t_0, t_1]$ and $p \geq q_0$.

For more details on the refined equilibrium with excess supply, see Ewerhart and Feubli (2012).

6.2 Unique refined equilibrium with excess supply

In this section we determine the conditions under which there is a unique refined equilibrium with excess supply.

Lemma 3. Assume $\partial\varepsilon^S/\partial p = 0$ for $q_0 < p \leq q_1$. Then, $p' = q_1$ is the unique refined equilibrium p_e^* with excess supply.

The proof of Lemma 3 is equal to the proof of Theorem 6 in Ewerhart and Feubli (2012) and is therefore omitted.
The intuition underlying Lemma 3 is as follows. With a constant elasticity of supply, the inflow of high-quality sellers does not change with price. Even though the fraction of inflowing sellers decreases, the quality of these sellers increases with price. Hence, the increase in average quality caused by an increase in price is such that the price-quality ratio remains the same, i.e., the price-quality ratio is constant for prices $q_0 < p \leq q_1$. Buyers therefore prefer higher over lower prices because higher prices imply a higher average quality at the same price-quality ratio. As a result, for $t > p'/\bar{q}(p')$, $t\bar{q}(p) - p$ is increasing in price for $q_0 < p \leq q_1$ and decreasing for $p > q_1$. For $t = p'/\bar{q}(p')$, $t\bar{q}(p) - p = 0$ for $q_0 < p \leq q_1$. Therefore, a buyer with $t \in [p'/\bar{q}(p'), t_1]$ wants to trade at $p = q_1$. Hence, no matter whether sellers or buyers unilaterally announce prices, the announced price will be $p = q_1$. Consequentially, there is a unique refined equilibrium with excess supply as described in Definition 4 if the price-setting mechanism results in $p' = q_1$ with $S(p') > D(p')$.

6.3 Comparative statics analysis

Consider two markets M_A and M_B, each as described in Sections 6.1 and 6.2. Suppose that the two intervals $[q_0, q_1]$ and $[t_0, t_1]$ are common to both markets. Market M_A is characterized by the initial vector $(N_A, J_A, F_A(.), H_A(.))$, whereas market M_B is described by the vector $(N_B, J_B, F_B(.), H_B(.))$. Assume that in both markets, $p' = q_1$, $S_i(q_1) > D_i(q_1)$ for $i = A, B$, and the condition in Lemma 3 is fulfilled. Hence, both markets feature a unique refined equilibrium price $p_e^* = q_1$ with excess supply.

Theorem 3. Assume that

7) $f_B(p)/F_B(p) \leq f_A(p)/F_A(p)$ for all $p > q_0$,

8) $J_B\,(1 - H_B(t)) \leq J_A\,(1 - H_A(t))$ for all t.

Then, $V_B^* \leq V_A^*$, where V_i^* denotes the trade volume in the refined equilibrium of market M_i for $i = A, B$.

Proof. We first show that a change in the elasticity of supply has no effect on the equilibrium price $p_e^* = q_1$. Given that the elasticity of supply is constant, $t\bar{q}(p) - p$ reads $p(t/c - 1)$ with $c = p/\bar{q}(p)$ and $t \in [p'/\bar{q}(p'), t_1]$ for $q_0 < p \leq q_1$, and $t\bar{q}(q_1) - p$ for $p > q_1$. From Lemma 2 we know that a decrease in the elasticity of supply implies an increase in $p/\bar{q}$, i.e., an increase in c. Thus, the slope of $t\bar{q}(p) - p$ decreases for $q_0 < p \leq q_1$, but the function is still increasing or constant for $q_0 < p \leq q_1$ and decreasing for $p > q_1$, where $t \in [p'/\bar{q}(p'), t_1]$. Hence, buyers with $t \geq p'/\bar{q}(p')$ still unilaterally announce $p = q_1$. By Definition 4, this suffices for the equilibrium price to remain $p_e^* = q_1$. Hence, in both markets M_A and M_B, $p_e^* = q_1$. According to Lemma 2, condition 7 is sufficient for $\bar{q}_B(p) \leq \bar{q}_A(p)$ for all $p > q_0$. Thus, the marginal buyer with $t = p_e^*/\bar{q}(p_e^*)$ is higher in market M_B than in market M_A. In combination with condition 8 it follows that $V_B^* = D_B(p_e^*) \leq D_A(p_e^*) = V_A^*$ with $p_e^* = q_1$. $\square$

The intuition of Theorem 3 relates to the intuition of Theorem 1. However, in the refined equilibrium with excess supply, the trade volume is solely determined by equilibrium demand $D_i(p_e^*)$ for $i = A, B$. Thus, for equilibrium trade volume to weakly decrease as one moves from market M_A to market M_B, the changes in the exogenous parameters are required to have a non-positive effect on demand. Thereby, the effect of these changes on supply remains irrelevant as long as there is still equilibrium excess supply. This is why in Theorem 3, there is no condition equivalent to condition 1 in Theorem 1. A non-positive reaction in demand can be achieved by either decreasing the size of the buyer population. Or it can be achieved by (additionally) decrease the aggregate willingness to buy and the elasticity of supply, i.e., the average quality of units.

Corollary 1. Assume that conditions 7 and 8 in Theorem 3 are satisfied such that $V_B^* \leq V_A^*$. If, in addition,

9) $N_B F_B(p) \geq N_A F_A(p)$ for all p,

$S_B(p_e^*) - V_B^* \geq S_A(p_e^*) - V_A^*$, where $S_i(p_e^*) - V_i^*$ denotes the equilibrium excess supply in market M_i for $i = A, B$.

Corollary 1 presents sufficient conditions for excess supply in the refined equilibrium to be larger in market M_B than in market M_A. That is, the difference between equilibrium supply and demand increases as one moves from market M_A to market M_B. Given that equilibrium demand is lower in market M_B than in market M_A, it suffices that the changes in the exogenous parameters are such that equilibrium supply is higher in market M_B than in market M_A. For supply's reaction to the exogenous changes to be non-negative, there are two possibilities. Either the exogenous changes are such that the size of the seller population increases. Or the changes (additionally) affect the quality distribution such that it is first-order stochastically dominated by the initial, thereby satisfying condition 7 in Theorem 3. Theorem 3 and Corollary 1 are illustrated in Figure V.

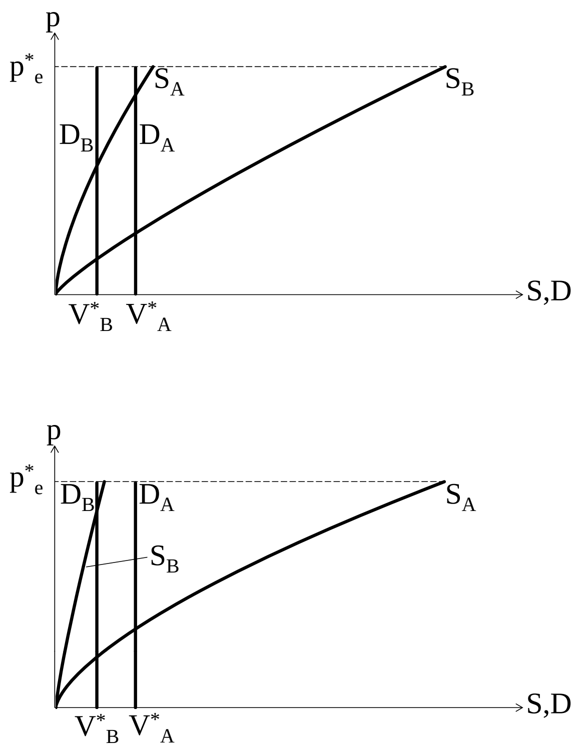

Figure V: Comparative statics of the refined equilibrium with excess supply

In Figure V, both markets M_A and M_B feature a power distribution for quality and a uniform distribution for buyer preferences. In both markets, the elasticity of supply is constant for $q_0 < p \leq q_1$ with $q_0 = 0$. Parameter values are chosen such that $S_i(q_1) - D_i(q_1) > 0$ for $i = A, B$.[10] Assuming that $p' = q_1$ in both markets, $p_e^* = q_1$ with $S_i(p_e^*) - V_i^* > 0$ for $i = A, B$. In the upper part of Figure V, all three conditions in Theorem 3 and Corollary 1 are fulfilled with strict inequality. Hence, equilibrium trade volume is lower in market M_B than in market M_A. Moreover, equilibrium excess supply is higher in market M_B than in market M_A. In the lower part of Figure V, only conditions 7 and 8 are satisfied, whereas condition 9 in Corollary 1 is violated. Given that the elasticity of supply and aggregate willingness to buy are still lower in market M_B, $V_B^* < V_A^*$. However, because the seller population

[10] In particular, $(N_A, J_A, F_A, H_A) = (20, 40, q^{1.5}, (t - 0.9)/1.3)$ and $(N_B, J_B, F_B, H_B) = (80, 20, q^{1.2}, (t - 0.9)/1.3)$, where $[q_0, q_1] = [0, 1]$ and $[t_0, t_1] = [0.9, 2.2]$.

is now much lower in market M_B than in market M_A, $S_B(p) < S_A(p)$ for all $p \geq q_0$. Under these circumstances, comparative statics results for excess supply are generally ambiguous. In the lower part of Figure V, parameter values are chosen such that equilibrium excess supply is lower in market M_B than in market M_A.[11]

A refined equilibrium with excess supply may also feature seller rationing. Ewerhart and Feubli (2012) define seller rationing in the standard set-up as follows. There is equilibrium seller rationing if among the sellers offering their good at p_e^*, some are able to sell their good and some are not, and some of the latter are not able to sell the good even if they announce a price $p < p_e^*$. Obviously, there may be seller rationing at $p_e^* = q_1$ with $S_i(p_e^*) > D_i(p_e^*)$ for $i = A, B$. Hence, if $S_B(p_e^*) - V_B^* \geq S_A(p_e^*) - V_A^*$, seller rationing in market M_B may be higher than seller rationing in market M_A. Note that since $p_e^* = q_1$, the equilibrium price in market M_B differs from the equilibrium price in market M_A if and only if q_1 differs across markets.

7 Empirical evidence

In this section, we illustrate our theoretical results for trade volume, price, and excess supply with empirical comparative statics studies of markets with asymmetric information.

An example for Theorem 2 is delivered by Genesove (1993). Genesove (1993) shows that used car dealers with a higher propensity to sell obtain a higher price in the wholesale used car auction. The wholesale used car auction, where dealers sell cars to other dealers, serves as a possibility to adjust the composition of stock. Individual quality of used cars and the dealers' stock composition are private information. However, individual propensity to sell used cars is observable. Intuitively, it is expected that dealers having more used cars than they want also sell some used cars of better quality, which in turn affects the price they get for their cars. To illustrate our results, we group high propensity dealers and denote them as market M_B. The group of low propensity dealers is denoted as market M_A. In Genesove's (1993) sample, 13.5 percent of used cars sold in the wholesale used car auction are offered by sellers with high propensity to sell. Because the size of the seller population in

[11] In particular, $N_A, J_A, F_A, H_A) = (80, 40, q^{1.5}, (t - 0.9)/1.3)$ and $(N_B, J_B, F_B, H_B) = (10, 20, q^{1.2}, (t - 0.9)/1.3)$, where $[q_0, q_1] = [0, 1]$ and $[t_0, t_1] = [0.9, 2.2]$.

market M_B is much lower than in market M_A, is seems reasonable to assume that supply in market M_B is lower than in market M_A at every price p. We expect the aggregate willingness to buy to be approximately the same in both markets because in the wholesale used car market, there is a single auction for both types of sellers. Using the propensity to sell as a measure for quality, average quality in market M_B exceeds average quality in market M_A. Hence, it seems that the wholesale used car auction delivers empirical evidence for Theorem 2.

Gibbons and Katz (1991) present a model which predicts that post-displacement wages of otherwise observationally equivalent workers differ according to the cause of displacement. Gibbons and Katz (1991) find empirical evidence that post-displacement wages for laid-off workers are lower than those for workers unemployed as a result of a plant closing. If the cause of displacement is indeed a signal for a worker's average ability, the study of Gibbons and Katz (1991) delivers another useful example for Theorem 2.

In his seminal paper, Akerlof (1970) discusses the problem that the elderly often have difficulties finding appropriate (non-compulsory) medical insurance. Based on statistical data for the U.S., he shows that the fraction of policy holders decreases with the age of applicants. Inspired by Akerlof's (1970) example, we use the Current Population Survey 2005 of the U.S. Census Bureau (presented in DeNavas-Walt, Proctor, and Lee, 2006, for example) to illustrate Theorem 1. Thereby, we do not claim our example to be robust. We interpret the insurance market as follows. On the demand side, there are insurers which are looking for policy holders in order to keep their business going. On the supply side, there are agents which offer to become policy holders. These agents differ in their reservation price, i.e., the maximum premium they are willing to pay for insurance. The individual reservation price depends on the expected expenditures on health, where expenditures on health are assumed to increase with age. Insurers cannot observe individual expected expenditures on health. They are only able to observe the age of the agents, i.e., average expected expenditures on health. Trade volume in this market is defined as the number of policy holders. Denote the group of agents aged 35 to 44 as market M_A and those aged 55 to 64 as market M_B. According to the Current Population Survey 2005, the number of policy holders, i.e., trade volume, is higher in market M_A than in market M_B. In particular, over the years 1987 to 1995, there were between 34692 and 43078 thousand people aged 35 to 44, and out of these, 28353 to 31441 thousand people had a private health insurance. In those years, there were between 20528 to 21641 thousand people aged 55 to 64 and 15735 to 17423 thousand people out of this group were privately insured. For the health insurance market to be indeed an illustration of Theorem 1, supply, the elasticity of supply, and the aggregate willingness to buy have to be lower in market M_B. Because the population size in market M_B is noticeably lower than in market M_A, it is reasonable to assume that supply is lower in market M_B than in market M_A for every premium. Since expenditures on health are assumed to increase with age, average expected expenditures on health are higher in market M_B than in market M_A for every premium. Thus, average quality and the elasticity of supply are both lower in market M_B. Moreover, we expect every insurer to participate in both markets. Thus, the aggregate willingness to buy, i.e., the aggregate willingness to accept agents as policy holders, should be roughly the same in both markets. To sum up, our example seems to satisfy the conditions in Theorem 1, and the health insurance market seems to deliver empirical evidence for our comparative statics result for equilibrium trade volume.

The market for private health insurance may also deliver empirical support for Corollary 1. Murtaugh, Kemper, and Spillman (1995) use U.S. data on long-term care insurance. They estimate that of the insurance applicants aged 65, between 12 and 23 percent would be rejected, whereas of those aged 75, between 20 and 31 percent would be rejected. Denote the group of applicants aged 65 as market M_A and those aged 75 as market M_B. The finding of Murtaugh, Kemper, and Spillman (1995) hints towards higher excess supply of potential policy holders in market M_B than in market M_A. Moreover, using age as an indicator for expected expenditures on long-term care, average expected expenditures are higher in market M_B. This implies that average quality and elasticity of supply in market M_B are lower than in market M_A. Thus, this empirical example seems to satisfy condition 7 in Theorem 3. However, because there are more applicants aged 65 than those aged 75, supply in market M_A seems to exceed supply in market M_B, which contradicts condition 9 in Corollary 1. So, this particular example may not be a perfect illustration of Corollary 1. Nevertheless, the example shows that the market for health insurance may be useful to find empirical evidence for Corollary 1.

8 Concluding remarks

This study has dealt with monotone comparative statics in markets characterized by asymmetric information. It offered conditions under which exogenous changes have predictable implications for volume and price in the unique market-clearing equilibrium. The analysis revealed that equilibrium trade volume declines if supply, elasticity of supply, and aggregate willingness to buy decrease. Clear-cut predictions of changes in the equilibrium price require a different set of conditions. In particular, the equilibrium price increases if supply decreases, and both the elasticity of supply and aggregate willingness to buy increase. By dropping the equilibrium condition of market-clearing, we were also able to discuss comparative statics of excess supply. We showed that equilibrium excess supply increases if supply increases, and both the elasticity of supply and aggregate willingness to buy decrease. By discussing empirical studies of comparative statics in markets with asymmetric information, we also delivered empirical support for our results.

Appendix

Lemma 2. Assume $\varepsilon_B^S(p) \leq \varepsilon_A^S(p)$ for all $p > q_0$. Then $\bar{q}_B(p) \leq \bar{q}_A(p)$ for all $p > q_0$.

Proof. Assume $\varepsilon_B(p) \leq \varepsilon_A(p)$ for $p > q_0$, i.e.,

$$\frac{\partial \log F_B}{\partial \log p} \leq \frac{\partial \log F_A}{\partial \log p}. \tag{A.1}$$

Since $\partial \log p = (1/p)\partial p$, this implies

$$\frac{\partial \log F_B}{\partial p} \leq \frac{\partial \log F_A}{\partial p}. \tag{A.2}$$

Integrating (A.2) over the interval $[p', p]$ for $q_0 < p' < p$ yields

$$\log F_B(p) - \log F_B(p') \leq \log F_A(p) - \log F_A(p'). \tag{A.3}$$

Applying the negative exponential function to both sides of inequality (A.3), we obtain

$$\frac{F_B(p')}{F_B(p)} \geq \frac{F_A(p')}{F_A(p)}. \tag{A.4}$$

By another integration over the interval $[q_0, p]$ one finds

$$\frac{1}{F_B(p)} \int_{q_0}^{p} F_B(q)dq \geq \frac{1}{F_A(p)} \int_{q_0}^{p} F_A(q)dq. \tag{A.5}$$

Using partial integration one can rewrite (A.5) as

$$q - \frac{\int_{q_0}^{p} q f_B(q)dq}{F_B(p)} \geq q - \frac{\int_{q_0}^{p} q f_A(q)dq}{F_A(p)}, \tag{A.6}$$

which in turn results in

$$\frac{\int_{q_0}^{p} q f_B(q)dq}{F_B(p)} \leq \frac{\int_{q_0}^{p} q f_A(q)dq}{F_A(p)} \tag{A.7}$$
$$\Leftrightarrow \bar{q}_B(p) \leq \bar{q}_A(p).$$

Hence, indeed, $\varepsilon_B(p) \leq \varepsilon_A(p)$ for $p > q_0$ implies $\bar{q}_B(p) \leq \bar{q}_A(p)$ for $p > q_0$. $\square$

References

Akerlof, G. (1970): "The Market for "Lemons": Quality Uncertainty and the Market Mechanism," *The Quarterly Journal of Economics*, 84, 488–500.

Athey, S. (2002): "Monotone Comparative Statics Under Uncertainty," *The Quarterly Journal of Economics*, 117, 187–223.

Daley, B., and B. Green (2010): "Waiting for News in the Market for Lemons," *Working Paper*.

DeNavas-Walt, C., B. D. Proctor, and C. H. Lee (2006): "Income, Poverty, and Health Insurance Coverage in the United States: 2005," Current Population Reports P60-231, U.S. Census Bureau.

Ewerhart, C., and P. Feubli (2012): "Unique Market-Clearing Equilibrium in Markets for "Lemons"," *Working Paper*.

Genesove, D. (1993): "Adverse Selection in the Wholesale Used Car Market," *Journal of Political Economy*, 101, 644–665.

Gibbons, R., and L. F. Katz (1991): "Layoffs and Lemons," *Journal of Labor Economics*, 9, 351–380.

Greenwald, B. C. (1986): "Adverse Selection in the Labour Market," *The Review of Economic Studies*, 53, 325–347.

Kessler, A. S. (2001): "Revisiting the Lemons Market," *International Economic Review*, 42, 25–41.

Kurlat, P. (2010): "Lemons, Market Shutdowns and Learning," *Working Paper*.

Leland, H. E. (1979): "Quacks, Lemons and Licensing: A Theory of Minimum Quality Standards," *Journal of Political Economy*, 87, 1328–1346.

Levin, J. (2001): "Information and the Market for Lemons," *RAND Journal of Economics*, 32, 657–666.

Mas-Colell, A., M. D. Whinston, and J. R. Green (1995): *Microeconomic Theory*. Oxford University Press New York and Oxford, first edn.

Murtaugh, C. M., P. Kemper, and B. C. Spillman (1995): "Risky Business: Long-Term Care Insurance Underwriting," *Inquiry*, 32, 271–284.

Rothschild, M., and J. Stiglitz (1976): "Equilibrium in Competitive Insuance Markets: An Essay on the Economics of Imperfect Information," *The Quarterly Journal of Economics*, 90, 629–649.

SPENCE, M. (1976): "Competition in Salaries, Credentials, and Signaling Prerequisites for Jobs," *The Quarterly Journal of Economics*, 90, 51–74.

VAN DEN BERG, G. (1994): "The Effects of Changes of the Job Offer Arrival Rate on the Duration of Unemployment," *Journal of Labor Economics*, 12, 478–498.

WILSON, C. (1980): "The Nature of Equilibrium in Markets with Adverse Selection," *The Bell Journal of Economics*, 11, 108–130.

Chapter 3

The Effect of Liquidity Injections on Interbank Money Markets

joint with Christian Ewerhart

1 Introduction

One of the most important goals of central banks is to ensure price stability, i.e., to keep the inflation rate at a low level. To achieve this goal, a central bank steers the interest rate level in interbank markets by using open market operations. To lower interbank market rates, the central bank conducts *reverse* open market operations to lend money to banks. If the central bank wants to raise interbank market rates, it uses open market operations to issue bonds.

The two main forms of reverse open market operations are fixed rate tenders and variable rate tenders. If the central bank offers credit through a fixed rate tender, it announces the interest rate the banks have to pay for one unit of credit and banks bid the amount of credit they wish to transact. If the central bank uses a variable rate tender, banks submit both the interest rate they are willing to pay and the amount of credit they wish to get. In a variable rate tender, the central bank orders the bids in a descending order and determines the market-clearing rate. Then, banks with bids above the market-clearing rate receive central bank credits and pay the market-clearing rate (multi-unit uniform-price auction) or their individual bid

(discriminatory multi-unit auction).
Central banks differ in the usage of fixed and variable rate tenders. For example, the Federal Reserve uses variable rate tenders to inject liquidity, whereas the Bank of England and the Swiss National Bank offer liquidity mainly through fixed rate tenders. The European Central Bank, in turn, switched between the two forms of reverse open market operations. Historical practice and doctrine may be the main drivers behind these preferences for a specific framework (see Bindseil, 2004). The Bank of England, for example, always targeted short-term interest rates. It seems natural that the Bank of England uses fixed rate tenders as a commitment to steer interbank market rates to levels around the tender rate. The Fed, in contrast, is more reluctant to take responsibilities for short-term rates, which makes tender rate fluctuations in variable rate tenders less problematic. The reason why the European Central bank switched from fixed rate tenders to variable rate tenders were not changes in doctrine, it was the behavior of participants. The demand for European Central Bank credits exceeded supply by far in fixed rate tenders which led to a very small allotment rate. To decrease demand for central bank credits to a suitable level, the European Central Bank decided to use variable rate tenders.
A natural question to ask is whether, besides doctrine and undesired allotment rates, there are other factors a central bank should take into account when choosing one of the two reverse open market operation frameworks. For example, there may be interbank market characteristics for which one framework is more suitable than the other.
In this paper, we focus on default and adverse selection as important features of interbank markets. In particular, we study an interbank market with asymmetric information on borrowers' probability of default. That is, borrowers know their individual probability of default, whereas lenders only observe the average default probability of borrowers in the market. Because lenders cannot distinguish borrowers with different default probabilities, all loans are settled at the same interbank rate. As a result, the market is characterized by adverse selection, i.e., an interest rate attracts only borrowers with relatively high probabilities of default. Therefore, interbank loans may be traded at a relatively high interbank rate. This in turn may induce the central bank to intervene. Observing the average default probability in the interbank market, the central bank may offer central bank credits as an alter-

native to interbank credits to keep the interbank rate close to some (lower) target rate.

In this set-up, our results on fixed and variable rate tenders are as follows. First, we find that in the variable rate tender (multi-unit uniform-price auction), there are multiple pure strategy equilibria. However, all these equilibria lead to the same outcome in the interbank market. Second, we show that both fixed and variable rate tenders decrease the interbank rate because they lower the demand for interbank credits. Third, it turns out that a variable rate tender may be the more effective instrument because, while injecting the same amount of liquidity, a variable rate tender may lower the interbank rate by more than a fixed rate tender. This result is due to the fact that the fixed rate tender also serves borrowers with relatively low default probabilities. For the relevant interest rates, the fixed rate tender lowers the demand for interbank credits therefore by less than a variable rate tender. As a result, the interbank rate is lower with a variable than with a fixed rate tender.[1] However, the fixed rate tender is the more flexible instrument because it not only allows to reproduce the variable rate tender's effect on the interbank rate. It also allows to fine-tune the interbank rate without the requirement to adjust the allotment amount. Fourth, we show that by lowering the interbank rate, both instruments increase social welfare, i.e., the sum of aggregate borrower, lender, and central bank rents. Thereby, the increase in the aggregate welfare of borrowers and lenders is larger with a fixed than with a variable rate tender. In contrast, the central bank's expected gains from lending minus the expected costs of an increase in the inflation rate are larger with a variable than with a fixed rate tender.

Our paper reveals that indeed, the central bank's choice of the type of open market operations should not only depend on doctrine and historical practice. It should also account for asymmetries in information on default probabilities and the resulting adverse selection. If the interbank market is characterized by asymmetric information on default probability, a variable rate tender may be preferable when large decreases in the interbank rate are required to reach the central bank's target. However, fixed rate tenders may be more appropriate if the interbank rate requires fine-tuning. Moreover, the central bank may prefer fixed rate tenders over variable rate tenders if it is interested in maximizing aggregate welfare of borrowers

[1] An intuitive discussion of this result has appeared in our IEW Working Paper No. 455.

and lenders. However, the central bank may want to use a variable rate tender if defaults on central bank credits are costly because they cause the inflation rate to increase.

The rest of the paper is organized as follows. Section 2 discusses related literature. The model is outlined in Section 3. In Section 4, we introduce fixed and variable rate tenders and study their effect on the interbank market rate. Section 5 discusses the effect of the monetary policy instruments on social welfare. Section 6 contains empirical evidence and a robustness analysis. Section 7 concludes.

2 Related literature

In this section we relate our paper to the literature and discuss selected contributions.

Our paper closely relates to Philippon and Skreta (2011), Tirole (2011), and House and Masatlioglu (2010). Philippon and Skreta (2011) study governmental lending in a credit market where there is asymmetric information on the default probability of the collateral. In this set-up, the government is able to induce a lower equilibrium interest rate if it competes with existing credit contracts by offering credits at a lower interest rate. Tirole (2011) and House and Masatlioglu (2010) show that governmental asset purchase programs increase the equilibrium asset price in a market with asymmetric information on the asset's return. The government can implement a higher asset price if it buys the assets with the lowest return (see Tirole, 2011). It can also implement a higher asset price if it fully replaces the demand side of the asset market and buys at a high price (see House and Masatlioglu, 2010). Because an asset can be interpreted as a claim on a counterparty,[2] asset purchase programs are similar to reverse open market operations, at least from a theoretical point of view. Our contribution to this literature is that we distinguish between different forms of lending programs and model the tenders as well as the behavior of tender participants.

While the above mentioned theoretical studies form a uniform picture of the effect of reverse open market operations on interbank rates, empirical findings are less

[2] See, for example, Tirole (2011).

homogeneous. Wu (2011) and McAndrews, Sarkar, and Wang (2008), for example, find that TAF (Term Auction Facility) helped to reduce the level of the Londoner Interbank Offered Rate (Libor).[3] Wu (2011) finds that TAF permanently reduced the Libor-OIS (overnight interbank swap) spread by 50 to 55 basispoints from 2007 to 2009. McAndrews, Sarkar, and Wang (2008) find a similar reduction in this spread from January 2007 to April 2008. This result, however, is contradicted by Taylor and Williams (2009). Running several regressions, they cannot find robust significant short-term and permanent effects of TAF on the Libor-OIS spread for the years 2007 and 2008.[4]

There is a strand of literature which discusses the two tender procedures in more detail. Thereby, these contributions focus less on the tenders' effect on the interbank rate but more on the bidding behavior of banks within these frameworks. Ayuso and Repullo (2003) show that there are multiple equilibria in the variable rate tender. Similar to our findings, these equilibria have in common that the marginal rate, i.e., the rate all successful bidders have to pay, equals the expected equilibrium interbank rate. Ayuso and Repullo (2003) show that this result does not change if the tender is discriminatory and bidders have to pay the interest rate they have submitted. Using a similar set-up, Catalão-Lopes (2010) confirms the multiplicity of equilibrium in the variable rate tender. In contrast to our set-up, these studies exclude adverse selection and use a dynamic set-up where central bank credits are provided before the interbank market opens. Empirical results on the banks' bidding behavior in variable rate tenders are delivered by Bindseil, Nyborg, and Strebulaev (2004) and Cassola, Hortaçsu, and Kastl (2011), for example. Using data on European Central Bank repo auctions from 2000 to 2001, Bindseil, Nyborg, and Strebulaev (2004) show that the bidding behavior of banks is affected by the banks' need for liquidity, the amount and costs of collateral, and expectations of future declines in the interbank rate. Cassola, Hortaçsu, and Kastl (2011) study data on European Central Bank auctions conducted in 2007. Their empirical findings support our results on the variable rate tender and therefore, we discuss their contribution more detailed

[3]The term auction facility was constructed as a variable rate tender and managed by the Federal Reserve. The final TAF was conducted on March 8, 2010.

[4]For the years 2007 to 2009, there exists evidence that the Libor was manipulated, i.e., the submissions for the Libor calculation were chosen such that the Libor turned out to be for the benefit of the manipulating banks. This finding may affect the empirical results on the Libor-OIS spread.

in Section 6.1.

There is also a line of literature which studies the banks' equilibrium bidding behavior in the fixed rate tender. This form of tender has been discussed mainly in the context of over- and underbidding. Overbidding (underbidding) refers to the phenomenon that banks submit very high (low) quantities, implying very low allotment ratios (full allotment). Over- and underbidding may occur because bidders expect the interbank rate to change in the near future (Bindseil, 2002), or bids do not have to be covered by collateral (Nautz and Oechssler, 2003), or the central bank injects less liquidity than needed because of its asymmetric loss function (Ayuso and Repullo, 2003), for example. In our study, massive over- and underbidding in the fixed rate tender is excluded because the borrower's demand for credit is exogenous.

The welfare-improving effect of open market operations has been discussed by Allen, Carletti, and Gale (2009), for example. In their paper, open market operations eliminate the interbank rate volatility which arises from the banks' uncertainty of future liquidity needs. This enables the interbank market to implement the constrained efficient allocation. In Heider, Hoerova, and Holthausen (2009), the central bank can provide liquidity at lower costs than the interbank market, which makes a central bank intervention desirable. However, Heider, Hoerova, and Holthausen (2009) point out that if the central bank fully replaces the interbank market, such interventions may also have a negative effect on social welfare because the market is no longer able to aggregate information and to monitor peers. The authors also present a short overview of the literature on rationales for central bank interventions.

Our paper also relates to the research on governmental interventions in dynamic set-ups. These studies differ from ours insofar as the informed market participants do not only differ in quality, but also in the trading delay they are willing to accept. In Chiu and Koeppl (2011), participants with low quality assets want to sell as fast as possible. Chiu and Koeppl (2011) show that in this case, a governmental asset purchase program may jumpstart the market if it offers sellers with low quality assets to immediately sell at a price which is attractive for them, but not for high quality sellers. Guerrieri and Shimer (2012) show that a subsidy for anyone selling low quality assets raises the price of these assets and increases trade volume of high quality assets.

3 Model

In this section we present the set-up of the unsecured interbank market. The set-up is kept close to the standard set-up of markets with asymmetric information.[5] Moreover, we define the market-clearing equilibrium, present conditions for its uniqueness, and perform comparative statics analysis.

3.1 Set-up

There is a continuum of price-taking and risk-neutral borrowers, each endowed with the same amount of cash L. Borrowers have the possibility to invest in an individual project. Every project costs $L + I$, where $I = 1$. In case the project succeeds, it returns X, and the probability of success is given by p. In case the project fails, it returns zero. The projects differ across borrowers insofar as $X(p)$ and p are individual for each borrower. We assume that $pX(p)$ strictly decreases with p, i.e., the higher the probability of success, the lower the expected return.[6] Because the borrowers' cash holdings do not cover the project costs, they need an interbank credit of size I. Borrowers pay $(1 + r)I$ to the lender if the project succeeds, where r is the interbank interest rate for one unit of credit. They default on the credit and repay nothing if the project fails. We assume the outside option of not borrowing and not investing in the project to generate zero utility. Thus, at interbank interest rate r, a borrower seeks credit in the interbank market if and only if

$$-L + p(X(p) - (1 + r)I) \geq 0. \tag{1}$$

Note that this implies $X(p) > (1 + r)I$ and $pX(p) > L$. A borrower is indifferent between the outside option of not investing and getting a credit and investing if

$$r = q(p) \equiv \frac{pX(p) - L}{pI} - 1. \tag{2}$$

For any interest rates $r > q(p)$, the borrower prefers the outside option, whereas for any interest rates $r \leq q(p)$, the borrower prefers to seek credit in the interbank

[5] For the standard set-up of markets with asymmetric information, see Akerlof (1970) and Wilson (1980).

[6] A similar set-up where expected project returns differ across borrowers can be found in Ewerhart and Feubli (2012a), for example.

market and to invest in the project. Hence, $q(p)$ is the borrower's reserve rate. The derivative of $q(p)$ with respect to p is negative such that the reserve rate strictly decreases with the probability of success. This results from the fact that the expected project return decreases, whereas the expected repayment to the lender increases with p. Hence, a borrower with high probability of success and therefore, a low probability of default, drops out of the interbank market at low interbank interest rates. To keep the notation simple, we will write q instead of $q(p)$ in what follows. We assume that the reserve rate q is distributed according to some strictly positive density f on $[q_0, q_1]$.[7] Thus, if r is the single market rate, demand for interbank credits is given by $D(r) = W \cdot (1 - F(r))$, where W is the size of the borrower population and $F(q)$ is the cumulative distribution function of q.

There is also a continuum of price-taking and risk-neutral lenders with individual cash surplus. Each of these lenders may offer a credit of size one in the interbank market. Lenders know the functional relationship between p, $X(p)$, and q, i.e., they know that $p = L/(X(p) - (1 + q)I)$. So, if a lender observes q, she knows p and $X(p)$. However, lenders cannot observe a borrower's individual reserve rate q. They are only able to infer the average reserve rate $\bar{q}(r) = E[q|q \geq r]$ from the market rate r. Let $\bar{q}(r) = q_1$ at $r = q_1$. Since lenders cannot distinguish borrowers with different probabilities of default, all credits are granted at the same interest rate. This, however, leads to adverse selection. At any interest rate r, only those borrowers with weakly higher reserve rates are in the market. Thus, an interest rate r only attracts borrowers with relatively high probabilities of default. The lender's expected profit from a loan at interest rate r is given by

$$\pi(r; t) = \overline{p}(r)rI - t(1 - \overline{p}(r))I, \tag{3}$$

where $I = 1$, $\overline{p}(r) = E[p|q \geq r]$, and t denotes the lender's individual hedging type. The lender is able to hedge against losses from lending in the interbank market because her cash surplus may exceed one, the size of the interbank loan. The parameter t measures the fraction of the loan which is not hedged (partial hedging). Hedging types are distributed according to some density h on $[t_0, t_1]$. Note that the lender's hedging type is private information. For the lender's expected profit to be

[7] Imposing an assumption on the distribution of q is equivalent to imposing the same assumption on the distribution of p.

non-negative at interest rate r,

$$r - t \cdot \frac{1 - \overline{p}(r)}{\overline{p}(r)} \geq 0. \tag{4}$$

Thereby, $(1 - \overline{p}(r))/\overline{p}(r)$ is the ratio of the average probability of default and the average probability of repayment and denotes the expected relative riskiness of a loan at interest rate r. Because borrowers with high probability of success drop out of the interbank market at low interest rates, $\overline{p}(r)$ is strictly decreasing and $(1-\overline{p}(r))/\overline{p}(r) = 1/\overline{p}(r) - 1$, i.e., the relative riskiness of a loan, is strictly increasing in r. A lender is willing to offer her funds as credit in the interbank market if $r - tg(\bar{q}(r)) \geq 0$, where $g(\bar{q}(r)) = 1/\overline{p}(r) - 1$ with $g(.)$ strictly increasing in $\bar{q}(r)$. We assume t_0 to be such that $r - t_0 g(\bar{q}(r)) > 0$ at $r = q_1$, i.e., the lender with the highest fraction hedged expects her utility from lending to be positive at the highest possible interest rate $r = q_1$. Define $H(t)$ as the fraction of lenders with hedging types $\leq t$. If t is the highest lender type which is willing to offer credit at r, the aggregate willingness to lend is described by $J \cdot H(t)$, i.e., the mass of lenders with a hedging type weakly below t. Thus, if r is the single market rate, supply is given by $S(r) = J \cdot H(r/g(\bar{q}(r)))$, where J is the size of the lender population.
We define a market-clearing equilibrium in the interbank market as follows.

Definition 1. A market-clearing equilibrium is an interest rate r^* for which $S(r^*) = D(r^*)$, i.e.,

$$J \cdot H\left(r^*/g(\bar{q}(r^*))\right) = W \cdot (1 - F(r^*)). \tag{5}$$

In our set-up, we exclude the possibility of moral hazard, i.e., the possibility that the bank misbehaves and invests low effort to prevent the failure of the project. Instead, we assume that each bank wants the project to succeed. The assumption builds on the observation that in our set-up, the effort invested to prevent the project's failure seems to be naturally high. This is because the bank invests its own equity L and therefore, it has some skin in the game.[8]

[8] This is in line with Kharroubi and Vidon (2009), for example. They find that banks pay particular attention to the success of a project and the moral hazard problem is mitigated if banks partially finance the project through equity.

3.2 Unique market-clearing equilibrium

To simplify the comparison of the two tenders and their effect on the interbank money market, we proceed with a unique market-clearing equilibrium. We thereby apply Ewerhart and Feubli's (2012c) results on the uniqueness of the market-clearing equilibrium in a goods market with asymmetric information on the good's quality. For the reader's convenience, we provide the details of this application.

Let $\varepsilon^D = (\partial D/\partial r) \cdot (r/D)$ denote the *elasticity of demand*. Moreover, let $\varepsilon^g = (\partial g/\partial \bar{q})/(\bar{q}/g)$ denote the *elasticity of relative riskiness*. The elasticity of relative riskiness gives the percentage change in the expected relative riskiness of a loan in response to a one percent increase in the average reserve rate.

Lemma 1. Assume $\partial|\varepsilon^D|/\partial r \geq 0$ and $\varepsilon^g \leq 1$ for $r \geq q_0$. Then, $\partial S(r)/\partial r \geq 0$, and the market-clearing equilibrium is unique.

Proof. The proof of Lemma 1 can be found in Appendix A1.

The intuition of Lemma 1 is as follows. For supply to be increasing in the interest rate, it has to be more attractive for lenders to offer their funds at higher interest rates. If the ratio of the interest rate to the riskiness $r/g(\bar{q}(r))$ is increasing in r, the payback from a loan increases by more than the expected riskiness of the loan. Therefore, lenders prefer higher over lower interest rates. An increasing ratio of the interest rate to the riskiness can be achieved by restrictions on the elasticity of demand and the elasticity of the relative riskiness. If the elasticity of demand is increasing in r, the percentage of borrowers who drop out of the interbank market increases with the interest rate. But this means that the "market share" of marginal borrowers, whose reserve rate is the lowest in the market, increases with r. Thus, even though the average reserve rate increases with r, it does not increase as much as the interest rate. Moreover, because the elasticity of the relative riskiness of loans is low, an increase in the average reserve rate leads only to a relatively small increase in the expected relative riskiness of loans. This is why lenders prefer higher over lower interest rates and supply is increasing in the interbank rate.

Figure I illustrates the unique interbank market equilibrium.

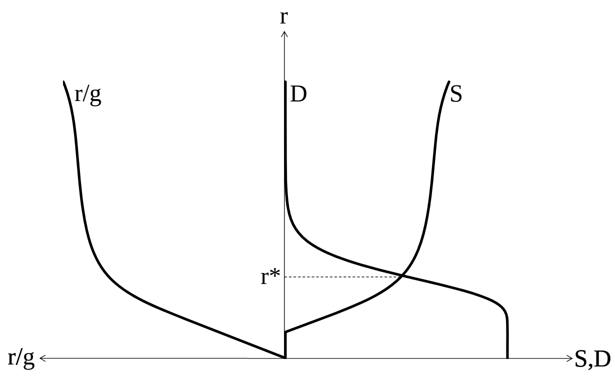

Figure I: Interbank market equilibrium

In Figure I, reserve rates are log-normally distributed, whereas hedging types are uniformly distributed, i.e.,

$$(W, J, f(.), H(.)) = \left(1, 1, \frac{e^{-\frac{\ln(q-0.6)^2}{0.1321}}}{0.257q\sqrt{2\pi}}, \frac{t-0.3}{0.95}\right).$$

Moreover, to keep the numerical example simple, we assume $L = 1$ and $E[X(p)|q \geq r] = 2(1 + \bar{q}(r))$. In particular, we assume that for a borrower with reserve rate q, the project return in case of success is twice the repayment to the lender at interest rate q, i.e., $X(p) = 2(1 + q)$. Thus, in the numerical example, $g(\bar{q}(r)) = \bar{q}(r)$.

3.3 Comparative statics of the equilibrium interbank rate

Before we introduce reverse open market operations, we conduct a comparative statics analysis of the interbank market-clearing equilibrium r^*. We thereby refer to the comparative statics results in Ewerhart and Feubli (2012b).

For expository reasons, we introduce two independent interbank markets M^A and M^B, each as described in Section 3.1. It is assumed that the intervals $[q_0, q_1]$ and $[t_0, t_1]$ are common to both interbank markets. We capture all exogenous parameter changes by assuming that market M^A is characterized by the initial vector $(W^A, J^A, F^A(.), H^A(.))$, whereas market M^B is characterized by the vector $(W^B, J^B, F^B(.), H^B(.))$, which may differ from vector $(W^A, J^A, F^A(.), H^A(.))$. The elasticity of demand is non-decreasing and the elasticity of relative riskiness is low in both markets such that they feature a unique market-clearing equilibrium (see Lemma 1).

The comparative statics analysis for these interbank markets has an important feature. As will be shown below, the average reserve rate is lower in market M^B than in market M^A for $r < q_1$, if for these interest rates, the elasticity of demand is higher in market M^B than in market M^A.[9]

Lemma 2. Assume $\left|\varepsilon^{D^B}(r)\right| \geq \left|\varepsilon^{D^A}(r)\right|$ for all $r < q_1$. Then $\bar{q}^B(r) \leq \bar{q}^A(r)$ for all $r < q_1$.

The proof of Lemma 2 is similar to the proof of Lemma 2 in Ewerhart and Feubli (2012b) and is left to Appendix A2. Intuitively, if the elasticity of demand is higher in market M^B than in market M^A for every interest rate $r < q_1$, the market share of marginal borrowers with the lowest reserve rate is higher in market M^B at these interest rates. Thus, the average reserve rate is everywhere lower in market M_B than in market M^A. Recall that $g(\bar{q}(r))$ is strictly increasing in $\bar{q}(r)$. Thus, $g(\bar{q}^B(r)) \leq g(\bar{q}^A(r))$ such that market M^B is less risky than market M^A.

We now turn to the comparative statics analysis of the equilibrium interbank rate r^*.[10]

Proposition 1. Assume that

1) $W^B\left(1 - F^B(r)\right) \leq W^A\left(1 - F^A(r)\right)$ for all $r \leq q_1$,

2) $f^B(r)/(1 - F^B(r)) \geq f^A(r)/(1 - F^A(r))$ for all $r < q_1$

3) $J^B H^B(t) \geq J^A H^A(t)$ for all t.

Then, $r^B \leq r^A$, where r^i denotes the respective interest rate in the unique market-clearing equilibrium in interbank market $M^i = M^A, M^B$.

[9]Ewerhart and Feubli (2012b) derive a similar result for goods markets and show that the average quality of goods is higher with a higher elasticity of supply.

[10]See Ewerhart and Feubli (2012b) for a similar comparative statics result for equilibrium price in goods markets.

Proof. The proof of Proposition 1 is left to Appendix A3.

Intuitively, for the market-clearing equilibrium interest rate to decrease as one moves from market M^A to market M^B, it suffices that the exogenous parameter changes have a non-positive effect on demand and a non-negative effect on supply. Consider the case where the size of borrower and lender populations are the same in both markets, i.e., $W^B = W^A$ and $J^B = J^A$. For the demand reaction to be non-positive, the reserve rate distribution has to change such that it first-order stochastically dominates the initial distribution (condition 1). In this case, the supply reaction is non-negative if two conditions are satisfied. Supply depends positively on both the distribution of lender types t and the expected relative riskiness of a loan, $g(\bar{q}(r))$. Thus, if there is an increase in the elasticity of demand (condition 2) and the willingness to lend weakly increases (condition 3), the supply reaction is non-negative. It follows that the conditions in Theorem 1 are sufficient for the equilibrium interbank rate to be unambiguously weakly lower in market M^B than in market M^A.

4 Reverse open market operations

We now introduce reverse open market operations through which a central bank offers $k > 0$ central bank credits of size one to interbank borrowers.

If a central bank decides to conduct a reverse open market operation, it announces terms and conditions for central bank credits as soon as the interbank market opens. The central bank knows the functional relationship between p, $X(p)$, and q, but only observes the type distributions F and H and cannot distinguish borrower types. For borrowers in the interbank market, a central bank credit is the alternative funding option which is chosen if it generates a utility as least as high as the utility from an interbank credit. Those borrowers who get a central bank credit do not seek credit the interbank market. Those borrowers who apply for a central bank credit but are denied may seek an interbank credit.

We now turn to the details of fixed and variable rate tenders and discuss their effect on the unique interbank market rate.

4.1 Fixed rate tender

If the central bank offers credit through a fixed rate tender, it announces the vector (k, r^I), where k denotes the amount of central bank credits and r^I denotes the interest rate it charges for a central bank credit. The interest rate r^I is the same for all borrowers. Each borrower may then apply for one central bank credit. If there are more than k borrowers applying for a central bank credit, the central bank *randomly* chooses k applicants and serves each with a credit.

We use standard definitions to describe demand and supply in the interbank market with a fixed rate tender. Residual demand is the difference between total demand and demand that has been supplied by the central bank. Hence, for any given r^I and k, residual demand equals

$$D^F(r; r^I, k) = \begin{cases} D(r) & \text{for } r < r^I \\ D(r) \cdot x & \text{for } r \geq r^I, \end{cases} \tag{6}$$

where $x = 1 - k/D(r^I)$. If the interbank market rate r is lower than r^I, all borrowers prefer interbank credits over central bank credits such that no borrower applies for a central bank credit. For interbank market rates $r \geq r^I$, demand for central bank credits is given by $D(r^I)$, and the fraction $k/D(r^I)$ is supplied with a central bank credit. Residual supply is given by

$$S^F(r; r^I, k) = J \cdot H\left(\frac{r}{g(\bar{q}^F(r))}\right) \tag{7}$$

for interest rates $r \leq q_1$, where $\bar{q}^F(r)$ is the residual average reserve rate in the market. We exclude the possibility that a reverse open market operation changes the distribution of t, i.e., the distribution of the lenders' hedging types.

The market-clearing equilibrium in the interbank market with a fixed rate tender is defined as follows.

Definition 2. Given the vector (r^I, k), a market-clearing equilibrium in the interbank market with a fixed rate tender is an interest rate r^F for which $S^F(r^F) = D^F(r^F)$.

Theorem 1. Assume $k \leq D(r^I) - S(r^I)$ and $r^I < r^*$. Then, r^F exists and $r^F < r^*$.

Proof. To prove Theorem 1, we will first show that r^F exists. In contrast to demand, there is a discontinuity in residual demand as a function of the interest rate. In particular, there is a discontinuous drop in residual demand at interest rate $r = r^I$, where it drops from $D^F(r) = D(r)$ to $D^F(r) = D(r) - k$. However, the definition of the market-clearing equilibrium requires that at r^F, residual demand and residual supply *intersect*. If $k \leq D(r^I) - S(r^I)$ and $r^I < r^*$, $D^F(r^I) \geq S^F(r^I)$ such that by Lemma 1, residual demand and residual supply indeed intersect and r^F exists.

We now prove that the effect of a fixed rate tender fulfills all three conditions in Proposition 1 such that $r^F \leq r^*$. Condition 1 in Proposition 1 requires that $D^F(r) \leq D(r)$ for all $r \leq q_1$. Because $k \leq D(r^I) - S(r^I)$, $x = 1 - k/D(r^I) < 1$. Then, from (6) it follows that indeed, $D^F(r) \leq D(r)$ for all $r \leq q_1$.

Condition 2 in Proposition 1 requires that $\varepsilon^{D^F}(r) \geq \varepsilon^D(r)$ for all $r < q_1$. For interest rates $r < r^I$, residual demand equals demand such that the elasticity of residual demand is the same as the elasticity of demand. For interest rates $r > r^I$, the elasticity of residual demand is defined as

$$\varepsilon^{D^F}(r) = \frac{\partial D^F(r)}{\partial r} \cdot \frac{r}{D^F(r)} = \frac{\partial D(r)x}{\partial r} \cdot \frac{r}{D(r)x} = \frac{\partial D(r)}{\partial r} \cdot \frac{r}{D(r)}. \tag{8}$$

At $r = r^I$, $\varepsilon^{D^F}(r) > \varepsilon^D(r)$. Thus, the elasticity of residual demand is the same as the elasticity of demand except at interest rate $r = r^I$, where it is higher than the elasticity of demand. Because residual supply depends on the residual average reserve rate, consider the following. Residual density of reserve rates is given by $f^F(q) = f(q)$ for $r < r^I$ and $f^F(q) = xf(q)$ for $r \geq r^I$. Hence, the probability that q is weakly higher than r reads $1 - F(r)$ for $r < r^I$ because at these interbank rates, borrowers do not participate in the fixed rate tender. However, for $r \geq r^I$, $(1 - F(r))\,x$. Thus, the residual average reserve rate is $\bar{q}^F(r) = \bar{q}(r)$ for $r < r^I$ and

$$\bar{q}^F(r) = \frac{x \int_r^{q_1} qf(q)dq}{x(1 - F(r))} = \frac{\int_r^{q_1} qf(q)dq}{1 - F(r)} = \bar{q}(r) \tag{9}$$

for $r \geq r^I$. Thus, the fixed rate tender has no effect on the average reserve rate. From this it follows that $g(\bar{q}^F(r)) = g(\bar{q}(r))$ for $r \leq q_1$.

Condition 3 requires that $JH^F(t) = JH(t)$ for all t. This condition is satisfied by

assumption. From the discussion of the residual average reserve rate it follows that $S^F(r) = J \cdot H(r/g(\bar{q}^F(r))) = J \cdot H(r/g(\bar{q}(r))) = S(r)$ for all interbank rates $r \geq q_0$. Hence, because all conditions in Proposition 1 are satisfied, $r^F \leq r^*$.
Finally, we prove that $r^F < r^*$. If $r^I \geq r^*$, $r^F = r^*$. This follows from $D^F(r) = D(r)$ and $S^F(r) = S(r)$ for $r \leq r^I$, which in turn follows from equations (6), (7), and the fact that $\bar{q}^F(r) = \bar{q}(r)$ for $r \leq r^I$. However, if $r^I < r^*$, $D^F(r) < D(r)$, whereas $S^F(r) = S(r)$ for $r \geq r^I$. Hence, $r^F < r^*$. Note that in case $k = D(r^I) - S(r^I)$, $r^F = r^I$, otherwise, $r^F > r^I$. $\square$

Intuitively, a fixed rate tender lowers the equilibrium interbank rate because it lowers demand for interbank credits at high interest rates. However, a fixed rate tender has no effect on supply of interbank credits. Even though some borrowers drop out of the interbank market at high interest rates, the distribution of reserve rates remains the same because the drop-out group is a *random* selection of borrowers with $q \geq r^I$. Therefore, there is no change in the ratio of the interest rate to the riskiness $r/g(\bar{q}(r))$ at high interest rates.
Theorem 1 describes a fixed rate tender with a set (r^I, k) such that interbank credits get cheaper, but are still more expensive than central bank credits. Figure II illustrates the equilibrium in the interbank market with and without a fixed rate tender.

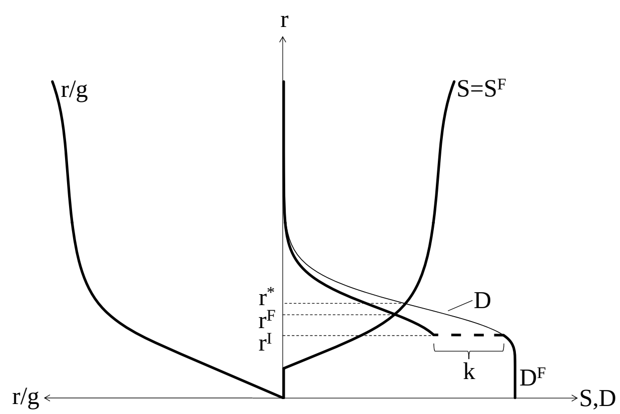

Figure II: Fixed rate tender: impact on interbank market rate

In Figure II, the interbank market without intervention is characterized by the same vector as the market in Figure I. Moreover, $(k, r^I) = (0.3, 1.2)$. For interest rates $r \geq r^I$, residual demand is proportional to demand because k randomly chosen borrowers with $q \geq r^I$ are able to use the central bank credit as an alternative funding option and drop out of the interbank market. This, however, does not change supply such that residual supply as a function of the interest rate equals supply in the market without intervention. Because there is a drop in demand for interbank credits at higher interest rates, the equilibrium interbank market rate declines. In the example, the equilibrium interbank market rate drops from $r^* = 1.79$ to $r^F = 1.59$.

4.2 Variable rate tender

Instead of a fixed rate tender, a central bank may also offer the same amount of central bank credits through a variable rate tender, i.e., a sealed-bid multi-unit uniform-rate auction. In this case, the central bank only announces k. All borrowers may then submit a bid, i.e., an interest rate they are willing to pay for a central bank credit. Denote the bid of the borrower with reserve rate q by b_q. It is assumed that borrowers choose the bid at random if they are indifferent between any available strategies. The individual bids are not publicly observable. The central bank collects the bids and sorts them in a descending order. The borrower with the highest bid is served first. Then, the central bank subsequently serves borrowers with lower bids until the central bank credits are exhausted. If there are several bidders who submit a bid equal to the kth highest bid, the central bank randomizes over these bidders. We define a successful bidder as a borrower who receives a central bank credit. All successful bidders pay the same interest rate r^T. This interest rate is set by the central bank after the borrowers submitted the bids and is such that k bids are weakly higher than r^T. We assume that borrowers anticipate the price-setting behavior of the central bank, i.e., they anticipate r^T before it is publicly observable. Borrowers who are denied a central bank credit may apply for credit in the interbank market. The unique market-clearing equilibrium in an interbank market with a variable rate tender is given by r^V. It is perfectly anticipated by the borrowers. We assume that $k < D(r^V)$.[11]

[11] I.e., central bank credits are scarce.

Definition 3 (Tender equilibrium). For any given r^V, the pure strategy equilibrium in the variable rate tender with scarce central bank credits is a strategy profile $b = (b_{q_0}, ..., b_{q_1})$ and a resulting interest rate r^T such that

1) b_q is the best response to all other borrowers' strategies b_{-q}
2) The mass of bidders with an equilibrium strategy $b_q \geq r^T$ equals k or is weakly higher than k if there are several bidders submitting the kth highest bid.

Denote the mass of successful bidders with $q \geq r$ as $(1 - x_V(r))D(r)$. For any given k and r^T, residual demand is given by

$$D^V(r; r^T, k) = \begin{cases} D(r) & \text{for } r < r^T \\ D(r) \cdot x_V(r) & \text{for } r \geq r^T. \end{cases} \tag{10}$$

Residual supply is given by

$$S^V(r; r^T, k) = J \cdot H\left(\frac{r}{g(\bar{q}^V(r))}\right) \tag{11}$$

for interest rates $r \leq q_1$, where $\bar{q}^V(r)$ is the residual average reserve rate in this market.
We define the market-clearing equilibrium in the interbank market with a variable rate tender as follows.

Definition 4 (Interbank equilibrium). Given the vector (k, r^T), the market-clearing equilibrium in the interbank market with a variable rate tender is an interest rate r^V for which $S^V(r^V) = D^V(r^V)$.

In the following, we first discuss the equilibrium in the variable rate tender (Definition 3). We then turn to the equilibrium in the interbank market with a variable rate tender (Definition 4) and compare it to the equilibrium in the interbank market without any intervention.

Theorem 2. For any given r^V, any equilibrium in the variable rate tender is such that

1) $r^T = r^V$

2) Only those borrowers with $q \geq r^V$ may submit $b_q \geq r^V$.

Proof. We prove Theorem 2 in four steps. First, the mass of borrowers bidding $b_q > r^V$ is smaller than k. If not, the kth highest bid is above r^V and $r^T > r^V$. Accordingly, each of the k successful bidders pays more for a central bank credit than for an interbank credit. Hence, for these bidders it is a profitable deviation to reduce the bid to r^V. This implies that r^T is at most r^V.
Second, the mass of borrowers bidding $b_q \geq r^V$ cannot be smaller than k. Otherwise, the kth highest bid is below r^V and $r^T < r^V$. This requires that because $D(r^V) > k$, some of the borrowers with $q \geq r^V$ submit a bid $b_q < r^T < r^V < q$. Accordingly, each of the k successful bidders pays less for a central bank credit than for an interbank credit. Hence, for those borrowers with $b_q < r^T < r^V < q$, it is a profitable deviation to raise the bid to $r^T \leq b_q < r^V \leq q$. This implies that r^T is at least r^V.
Third, it follows that the only possible candidate for an equilibrium as described in Definition 3 is $r^T = r^V$.
Fourth, $r^T = r^V$ is indeed an equilibrium and only those borrowers with $q \geq r^V$ may submit $b_q \geq r^V$. If $r^T = r^V$, central bank credits are as expensive as interbank credits. Accordingly, any borrower with $q \geq r^V$ is indifferent between being a successful bidder and submitting a losing bid. Hence, these borrowers may choose any bid because each bid leads to the same profit. Having chosen a bid, there is no profitable deviation if $r^T = r^V$. The probability to be a successful bidder is positive if a borrower submits a bid $b_q \geq r^V$. Hence, borrowers with $q < r^V$ will not submit a bid $b_q \geq r^V$ because for these borrowers it is not profitable to pay r^V. Hence, these borrowers may choose any bid below r^V, and each of these bids leads to the same profit. Having chosen a bid below r^V, there is no profitable deviation if $r^T = r^V$. $\square$

Figure III helps to gain an intuition for Theorem 2.

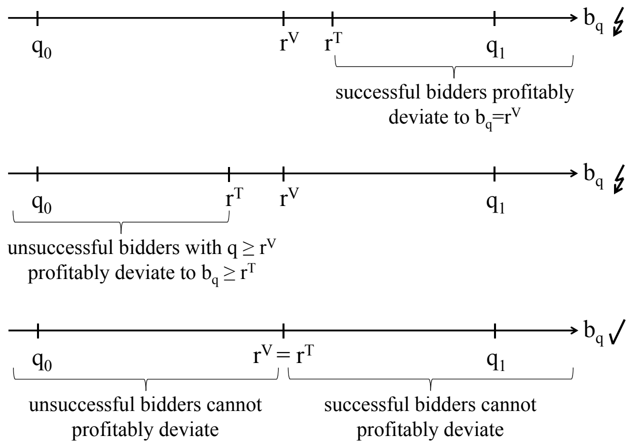

Figure III: Equilibrium in the variable rate tender

As illustrated in Figure III, any $r^T > r^V$ cannot be an equilibrium because successful bidders are not willing to pay more for a central bank credit than for an interbank credit and therefore, they have an incentive to deviate from their strategy. Any $r^T < r^V$ cannot be an equilibrium because borrowers with $q \geq r^V$ and $b_q < r^T$ want to be successful bidders and therefore, they have an incentive to deviate from their strategy.[12]

The equilibrium in the variable rate tender as described in Theorem 2 is not unique. There is a continuum of strategy profiles $b = (b_{q_0}, ..., b_{q_1})$ satisfying Definition 3 and the conditions in Theorem 2. The multiplicity of equilibrium results from the fact that if $r^T = r^V$, borrowers are indifferent between central bank and interbank credits. Fortunately, all these equilibria have in common that they lead to the same outcome in the interbank market with a variable rate tender. In the following, we will first discuss one specific variable rate tender equilibrium and the resulting interbank market rate r^V. Then, at the end of this section, we will show that all variable rate tender equilibria described in Definition 3 and Theorem 2 lead to this particular level of r^V.

[12]The variable rate tender differs from standard multi-unit uniform-price auctions insofar as in the variable rate tender, the borrowers' bids not only depend on their individual reserve rate, but also on the interest rate r^V, i.e., the interest rate for an interbank credit. For a comprehensive analysis of standard multi-unit uniform-price auctions, see Krishna (2010), for example.

Among the equilibria which are compatible with Theorem 2, we single out a particularly plausible one and describe it in Lemma 4.

Lemma 4. For a given r^V, one equilibrium in the variable rate tender is such that $r^T = r^V$, and borrowers with $q < r^V$ submit a bid $b_q = q$, whereas borrowers with $q \geq r^V$ submit the bid $b_q = r^V$.

Proof. Follows from the proof of Theorem 2.

In the equilibrium described in Lemma 4, $D(r^V)$ borrowers submit a bid $b_q = r^V$. Thus, residual demand is given by (10) with $x_V = 1 - k/D(r^V)$. The reason for $D^V(r) = D(r)$ at interest rates $r < r^V$ is that borrowers anticipate r^T. If, for some reason, the interest rate in the interbank market with a variable rate tender lies below r^T, borrowers prefer interbank credits over central bank credits.

Theorem 3. Given $k < D(r^V)$ and the tender equilibrium described in Lemma 4, $r^V < r^*$.

Proof. To prove Theorem 3, we will first prove that all three conditions in Proposition 1 are fulfilled and $r^V \leq r^*$. Condition 1 in Proposition 1 requires that $D^V(r) \leq D(r)$ for all $r \leq q_1$. Because $k < D(r^V)$, $x_V = 1 - k/D(r^V) < 1$. Then, from (10) it follows that indeed, $D^V(r) \leq D(r)$ for all $r \leq q_1$.
Condition 2 in Proposition 1 requires that $\varepsilon^{D^V}(r) \geq \varepsilon^D(r)$ for all $r < q_1$. For interest rates $r < r^V$, residual demand equals demand such that the elasticity of residual demand equals the elasticity of demand. For interest rates $r > r^V$,

$$\varepsilon^{D^V} = \frac{\partial D^V(r)}{\partial r} \cdot \frac{r}{D^V(r)} = \frac{\partial D(r) x_V}{\partial r} \cdot \frac{r}{D(r) x_V} = \frac{\partial D(r)}{\partial r} \cdot \frac{r}{D(r)}. \qquad (12)$$

At $r = r^V$, $\varepsilon^{D^V}(r) > \varepsilon^D(r)$. Thus, the elasticity of residual demand is the same as the elasticity of demand except at $r = r^V$, where it is higher than the elasticity of demand. Because residual supply depends on the residual average reserve rate, consider the following. The residual average reserve rate is $\bar{q}^V(r) = \bar{q}(r)$ for interest rates $r < r^V$ because for these interest rates, residual density of reserve rates is given by $f^V(r) = f(r)$. For interest rates $r \geq r^V$,

$$\bar{q}^V(r) = \frac{x_V \int_r^{q_1} q f(q) dq}{x_V (1 - F(r))} = \frac{\int_r^{q_1} q f(q) dq}{1 - F(r)} = \bar{q}(r) \qquad (13)$$

because for these interest rates, $f^V(r) = x_V f(r)$. Thus, a variable rate tender as described in Lemma 4 has no effect on the average reserve rate. From this it follows that $g(\bar{q}^V(r)) = g(\bar{q}(r))$ for all $r \leq q_1$.

Condition 3 requires that $JH^V(t) = JH(t)$ for all t. This condition is satisfied by assumption. From the discussion of the residual average reserve rate it follows that residual supply equals supply, i.e., $S^V(r) = J \cdot H(r/g(\bar{q}^V(r))) = J \cdot H(r/g(\bar{q}(r))) = S(r)$ for all $r \geq q_0$. Since all conditions in Proposition 1 are satisfied, $r^V \leq r^*$.

We now prove that $r^V < r^*$. From (10) follows that $D^V(r^V) = D(r^V) - k$. Because $S^V(r) = S(r)$ for all $r \geq q_0$, $k = D(r^V) - S(r^V)$ such that $D(r^V) > S(r^V)$. Because $D(r) - S(r)$ is strictly decreasing in r by Lemma 1, $r^* > r^V$. $\square$

Intuitively, a variable rate tender lowers the equilibrium interbank market rate because this form of intervention lowers demand for interbank credit at high interest rates. However, similar to a fixed rate tender, a variable rate tender has no effect on supply of interbank credits. This is because it leaves the distribution of borrower types in the market and therefore, the interest rate default cost ratio $r/g(\bar{q}(r))$, unaffected. Theorem 3 describes the simultaneous equilibrium with a variable rate tender where the tender equilibrium is best response to the interbank equilibrium and vice versa.

Figure IV illustrates the interbank market with and without a variable rate tender as described in Lemma 4.

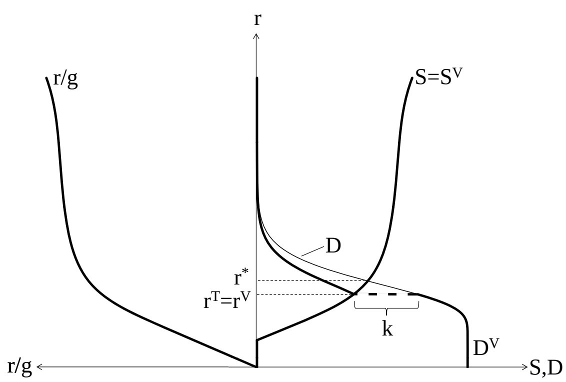

Figure IV: Variable rate tender: impact on interbank market rate

The interbank market without intervention in Figure IV is characterized by the same parameter vector as the interbank market in Figure I. Moreover, the central bank offers the same amount of credits as with a fixed rate tender, i.e., $k = 0.3$. For interest rates $r \geq r^V$, residual demand is proportional to demand because k randomly chosen borrowers with $q \geq r^V$ receive a central bank credit and drop out of the interbank market. Since there is a drop in demand at higher interest rates and supply remains the same, $r^V < r^*$ with $D(r^V) - k = S(r^V)$. In the example, the equilibrium interbank rate drops from $r^* = 1.79$ to $r^V = 1.52$.

Theorem 4. Any tender equilibrium described in Theorem 2 leads to the same interbank equilibrium r^V as the tender equilibrium described in Lemma 4.

Proof. Any equilibrium in Theorem 2 leads to a selection of successful bidders which is a random selection of borrowers with $q \geq r^V$. As shown in the proof of Theorem 2, borrowers with $q \geq r^V$ choose their bid randomly and at least k borrowers bid $b_q \geq k$ in the equilibrium. Moreover, borrowers with $q < r^V$ choose randomly among bids $b_q < r^V$. Thus, the ordering of the bids $b_q \geq r^V = r^T$ is a random order of borrowers with $q \geq r^V$. This is also the case if, by coincidence, all or some bidders with $q \geq r^V$ submit the same bid. Therefore, the successful bidders are always a random selection of borrowers with $q \geq r^V$.

It follows that for any equilibrium in Theorem 2, residual demand in the interbank market with a variable rate is given by (10). To see why, consider the following. If the group of successful bidders is a random selection of borrowers with $q \geq r^V$, the probability that a successful bidder has a reserve rate $q \geq r$ is given by $\left(k/D(r^V)\right)(1 - F(r))$. Thus, the probability that there are borrowers with $q \geq r$ in the interbank market equals $(1 - F(r))x_V$ for interest rates $r \geq r^V$, where $x_V = 1 - k/D(r^V)$. For interbank rates $r < r^V$, borrowers prefer interbank over central bank credits. Hence, residual demand is indeed given by (10).

For any equilibrium in Theorem 2, residual supply in the interbank market with a variable rate tender equals supply in the interbank market without any intervention. This follows from the proof of Theorem 3. In this proof we show that given residual demand (10), $\bar{q}^V(r) = \bar{q}(r)$ for all interbank rates $r \leq q_1$ such that $S^V(r) = S(r)$. Since residual demand for any equilibrium in Theorem 2 is given by (10), residual supply indeed equals supply in the interbank market without intervention.

It follows that any equilibrium in Theorem 2 leads to the same residual demand

and supply function as the equilibrium in Lemma 4. This in turn implies that any equilibrium in Theorem 2 leads to the same level of r^V as the equilibrium in Lemma 4. $\square$

4.3 Comparison: Effect on the equilibrium interbank rate

In this section we compare the effect of the two tenders described in Section 4.1 and Section 4.2 on the interbank rate. We assume that k is the same for both tenders.

Theorem 5. If r^I is such that $k = D(r^I) - S(r^I)$, $r^F = r^V$. If r^I is such that $k < D(r^I) - S(r^I)$, $r^F > r^V$.

Proof. If r^I is such that $k = D(r^I) - S(r^I)$, $D^F(r^I) = D(r^I) - k = S(r^I) = S^F(r^I)$, where the last equality follows from the proof of Theorem 1. Thus, $r^I = r^F = r^V$, where the latter equality comes from the fact that at $r = r^V$, $D^V(r) = D(r) - k$. However, if $k < D(r^I) - S(r^I)$, $D^F(r^I) = D(r^I) - k > S(r^I)$ and $r^F > r^I$. Thus, by Lemma 1, $r^V > r^I$ because at r^V, $D^V(r^V) = D(r^V) - k = S(r^V) = S^V(r^V)$. For interest rates $r > r^I$, $D^F(r) = D(r) \cdot x = D(r) - D(r)/D(r^I) \cdot k$, where $D(r)/D(r^I) < 1$ for $r > r^I$. Therefore, at r^V, $D^V(r^V) = D(r^V) - k = S(r^V)$, whereas $D^F(r^V) = D(r^V) - D(r^V)/D(r^I) \cdot k > D(r^V) - k = S(r^V)$. Thus, by Lemma 1, $r^F > r^V$. $\square$

Thus, while injecting the same amount of liquidity, the variable rate tender lowers the equilibrium interbank market rate by more than a fixed rate tender if r^I is low. The intuition is simple. If $r^I < r^V$, the fixed rate tender also serves borrowers with lower reserve rates, i.e., those with $r^I \leq q \leq r^V$. These borrowers do not get a central bank credit in the variable rate tender. Thus, with a fixed rate tender, less borrowers with $q \geq r^V$ drop out of the market than with a variable rate tender such that at higher interest rates, demand for interbank credits is larger with a fixed than with a variable rate tender. As a result, the interbank market rate may be lower with a variable than with a fixed rate tender, as illustrated in Figure V.

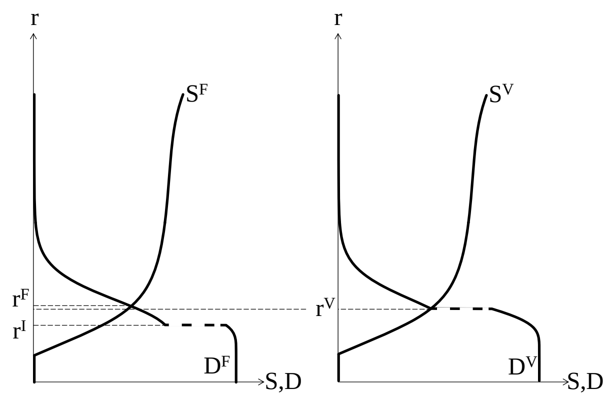

Figure V: Comparison of fixed and variable rate tender

We now briefly discuss the distance between r^F and r^V and its dependence on the primitives of the model. Assume r^I to be such that $k < D(r^I) - S(r^I)$ and thus, $r^V < r^F$ by Theorem 5. By Definition 4, $D^V(r^V) - S(r^V) = 0$, i.e., excess residual demand with a variable rate tender equals zero at interest rate r^V. However, from the proof of Theorem 5 we know that $D^F(r^V) - S(r^V) > 0$, i.e., residual excess demand with a fixed rate tender is positive at r^V. For interest rates $r > r^V$, excess residual demand $D^F(r) - S(r)$ goes to zero as the interest rate increases (Lemma 1). By Definition 2, it eventually reaches zero at interest rate $r = r^F$, where $D^F(r^F) - S(r^F) = 0$. Hence, the distance between r^F and r^V depends on the size of residual excess demand at $r = r^V$ and its decline at interest rates $r \geq r^V$. The smaller the residual excess demand at r^V and the larger its derivative with respect to r for interest rates $r \geq r^V$, the closer is r^F to r^V. Residual excess demand at $r = r^V$ is given by

$$D^F(r^V) - S(r^V) = D(r^V)\left(1 - \frac{k}{D(r^I)}\right) - S(r^V). \tag{14}$$

Residual excess demand at r^V is small if r^I is high and close to r^V. Then, $D(r^I)$ and therefore, $D^F(r^V) = D(r^V)(1 - k/D(r^I))$ is small. The derivative of residual excess demand reads

$$\begin{aligned}\frac{\partial}{\partial r}(D^F(r) - S(r)) &= \frac{\partial D(r)}{\partial r}\left(1 - \frac{k}{D(r^I)}\right) - \frac{\partial S(r)}{\partial r} \\ &= -f(r)W\left(1 - \frac{k}{D(r^I)}\right) - J \cdot h\left(\frac{r}{g(\bar{q}(r))}\right)\frac{\partial}{\partial r}\frac{r}{g(\bar{q}(r))} \qquad (15)\end{aligned}$$

for interest rates $r \geq r^V$. It is large for interest rates $r \geq r^V$ if the elasticity of demand is large, the elasticity of relative riskiness is small, and the elasticity of the willingness to lend is sufficiently large. To see why, consider the following. If the elasticity of demand, i.e., $|\varepsilon^D| = f(r)r/(1 - F(r))$, is large at r, $f(r)$ and the first term in (15) are large as well. However,

$$\frac{\partial}{\partial r}\frac{r}{g(\bar{q}(r))} = \frac{1}{g(\bar{q}(r))}\left(\bar{q}(r) - \varepsilon^g \frac{\partial \bar{q}(r)}{\partial r} r\right). \qquad (16)$$

If the elasticity of demand is large, $\bar{q}(r)$ and $g(\bar{q}(r))$ are low whereas $\partial \bar{q}(r)/\partial r$ is high such that, ceteris paribus, (16) can be either large or small. If the elasticity of relative riskiness, ε^g, is small, it is more likely that (16) is large. This, however, is not sufficient for the second term in (15) to be large. To make sure that this term is large, the elasticity of the willingness to lend, i.e., $h(t)t/H(t)$ has to be sufficiently large for $t = r/g(\bar{q}(r))$ such that $h(r/g(\bar{q}(r)))$ is sufficiently large at r. To sum up, if the fixed rate tender is such that $r^F > r^V$, r^F is close to r^V if r^I is close to r^V, and for interest rates $r \geq r^V$, the elasticity of demand and the elasticity of the willingness to lend are large, and the elasticity of relative riskiness is small.

Even though the variable rate tender may lower the interbank market rate by more than a fixed rate tender, the fixed rate tender is the more flexible reverse open market operation than the variable rate tender. This is because with a variable rate tender, the resulting equilibrium interbank rate is determined solely by k. However, with a fixed rate tender, the resulting equilibrium interbank rate is determined both by k and r^I. Hence, the fixed rate tender not only allows to implement the equilibrium in a market with a variable rate tender. The possibility to set the interest rate for central bank credits also allows some fine-tuning of the interbank rate which is not possible with the variable rate tender.

5 Social welfare

In this section we turn to the question how fixed and variable rate tenders affect social welfare. The answer to this question is not obvious at first sight because a decrease in the interbank market rate has two opposing effects on social welfare. On the one hand, borrowers' welfare increases because they have to pay less for a credit. On the other hand, lenders' welfare decreases because they face a lower ratio of the interest rate to the riskiness of a loan. Moreover, the implementation of a lower equilibrium interbank rate through reverse open market operations may be costly. In the following analysis, we will study the sign and size of the aggregate of these effects.

Definition 7. Equilibrium social welfare in the interbank market without any intervention is given by

$$U = U_B + U_L = \int_{r^*}^{q_1} D(r)dr + \int_{r_0}^{r^*} S(r)dr, \tag{17}$$

where U_B is the aggregate borrower welfare, U_L denotes the aggregate lender welfare, and r_0 is such that $r_0/g(\bar{q}(r_0)) = t_0$.

The equilibrium social welfare in the interbank market without any intervention is the sum of borrower and lender rents. In the interbank equilibrium, a borrower with reserve rate $q \geq r^*$ receives an interbank credit at rate r^* such that her individual rent equals $q - r^*$. Thus, the aggregate equilibrium welfare of borrowers reads

$$U_B = W \int_{r^*}^{q_1} (q - r^*) f(q) dq = W \int_{r^*}^{q_1} (1 - F(q))\, dq, \tag{18}$$

which is nothing else but $\int_{r^*}^{q_1} D(r)dr$.[13] A lender with hedging type $t \leq r^*/g(\bar{q}(r^*))$ lends her funds in the interbank market at rate r^*. This lender is willing to accept the ratio of the interest rate to the riskiness $r_t/g(\bar{q}(r_t))$, where r_t is such that $r_t - tg(\bar{q}(r_t)) = 0$. However, the lender gets $r^*/g(\bar{q}(r^*))$. Thus, her rent equals $r^*/g(\bar{q}(r^*)) - t$. Using a different measuring scale, lender t's rent can be written as

[13]For the second equality in (18) see Van den Berg (1994), for example.

$r^* - r_t$. The aggregate equilibrium welfare of lenders is therefore given by

$$U_L = J \int_{t_0}^{r^*/g(\bar{q}(r^*))} \left(\frac{r^*}{g(\bar{q}(r^*))} - t \right) h(t)dt = J \int_{t_0}^{r^*/g(\bar{q}(r^*))} H(t)dt, \tag{19}$$

which can be rewritten as $\int_{r_0}^{r^*} S(r)dr$. For illustration, see the left-hand side of Figure VI below.

5.1 Effect of the fixed rate tender on social welfare

Assume that the central bank uses a fixed rate tender to inject liquidity, and this fixed rate tender is characterized by (r^I, k), where $r^I < r^*$ and $k < D(r^I) - S(r^I)$. From Theorem 1 follows that $r^F < r^*$.

We assume that the social welfare in an interbank market with a fixed rate tender not only consists of the aggregate borrower and lender rents, but also of the central bank's returns and costs of lending k credits. If a borrower defaults on a central bank credit, the central bank has no direct default costs. This is because in principle, the central bank can replace any loss by simply printing money. However, printing money is costly because it may raise the inflation rate. Thus, the central bank's expected costs of default are the expected costs of a higher inflation rate. The central bank's expected profit from a loan at interest rate r is therefore assumed to be given by

$$\pi_c(r; t_c) = \overline{p}(r)r - t_c(1 - \overline{p}(r)), \tag{20}$$

where t_c are the costs of a higher inflation rate. If t_c is small, printing money is cheap because either a higher inflation rate is not very costly or printing money leads to a small increase in the inflation rate. We assume t_c to be low enough for the central bank's expected profit from a loan to be positive at interest rate r^I, i.e., $\overline{p}(r^I)r^I - t_c(1 - \overline{p}(r^I)) > 0$.

Definition 8. Equilibrium social welfare in the interbank market with a fixed rate tender is given by

$$
\begin{aligned}
U^F &= U_B^F + U_L^F + U_C^F \\
&= x \int_{r^F}^{q_1} D(r)dr + (1-x) \int_{r^I}^{q_1} D(r)dr + \int_{r_0}^{r^F} S(r)dr + k\left(r^I - t_c g(\bar{q}(r^I))\right),
\end{aligned}
\tag{21}
$$

where U_C^F is the central bank's welfare from a fixed rate tender.

With a fixed rate tender, the fraction x of borrowers with $q \geq r^F$ gets an interbank credit at interest rate r^F and has an individual rent of $q - r^F$. The fraction $1-x$ of borrowers with $q \geq r^I$ gets a central bank credit and pays the interest rate r^I. The individual rent of these borrowers is given by $q - r^I$. Lenders with $t \leq r^F/g(\bar{q}(r^F))$ lend their funds at the ratio of the interest rate to the riskiness $r^F/g(\bar{q}(r^F))$ such that their individual rent reads $r^F - r_t$. Moreover, the central bank lends k credits to randomly chosen borrowers with $q \geq r^I$ at the interest rate r^I. The aggregate welfare of borrowers and lenders in an interbank market with a fixed rate tender is illustrated on the right-hand side of Figure VI.

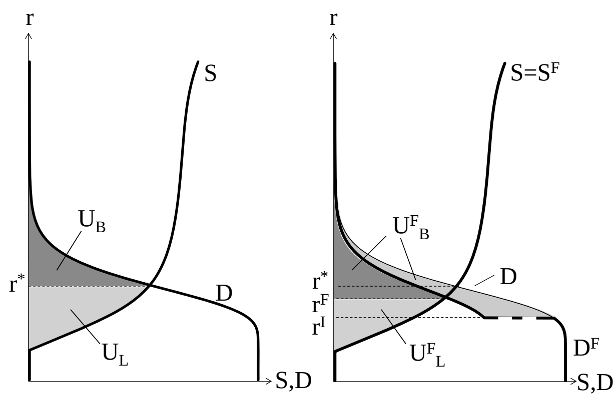

Figure VI: Social welfare in the interbank market without and with a fixed rate tender

Theorem 6. Assume $k \leq D(r^I) - S(r^I)$ and $r^I < r^*$ such that $r^F < r^*$. Then, $U^F > U$.

Proof. For the proof of Theorem 6, see Appendix A4.

Intuitively, the fixed rate tender increases social welfare because it increases the borrowers' aggregate welfare by more than it decreases the lenders' aggregate welfare. Moreover, by assumption, the expected costs of higher inflation are lower than the expected returns from central bank lending. Consider the effect of a fixed rate tender on the rents of those borrowers and lenders who are in the market at r^*. With a fixed rate tender, the borrowers with $q \geq r^*$ remain in the market at r^F and gain at least $r^* - r^F$. Thus, the aggregate gain in the rents of these borrowers equals at least $D(r^*)(r^* - r^F)$. From the lenders which are in the market at r^*, only those with $t \leq r^F/g(\bar{q}(r^F))$ remain in the market. These lenders lose $r^* - r^F$. The lenders with $t \in [r^F/g(\bar{q}(r^F)), r^*/g(\bar{q}(r^*))]$ drop out of the market. However, their loss equals $r^* - r_t$, where $r_t \in [r^F, r^*]$. Thus, the drop in the lenders' aggregate welfare is lower than $S(r^*)(r^* - r^F) = D(r^*)(r^* - r^F)$. From this it follows that the rise in borrower's aggregate welfare indeed overcompensates the drop in lenders' aggregate welfare. Thereby, the gain of those borrowers which enter the market because of a lower interest rate and the gain of those borrowers which only have to pay $r^I < r^F$ has not even been taken into account yet.

5.2 Effect of the variable rate tender on social welfare

Assume now that the central bank uses a variable rate tender to lend k central bank credits to banks. Suppose that the equilibrium in the variable rate tender is as described in Lemma 4 and $r^T = r^V$. From Theorem 3 it follows that $r^V < r^*$.

Definition 9. Equilibrium social welfare in the interbank market with a variable rate tender is given by

$$\begin{aligned} U^V &= U_B^V + U_L^V + U_C^V \\ &= \int_{r^V}^{q_1} D(r)dr + \int_{r_0}^{r^V} S(r)dr + k\left(r^V - t_c g(\bar{q}(r^V))\right), \end{aligned} \tag{22}$$

where U_C^V is the central bank's welfare from a variable rate tender.

We assume t_c to be low enough for the central bank's profit on a loan to be positive, i.e., $r^V - t_c g(\bar{q}(r^V)) > 0$.

Theorem 7. Assume $k < D(r^V)$ and the tender equilibrium to be as described in Lemma 4 such that $r^V < r^*$. Then, $U^V > U$.

Proof. The proof of Theorem 7 can be found in Appendix A5.

The effect of a variable rate tender on social welfare is similar to the effect of a fixed rate tender. A variable rate tender increases social welfare because it increases the borrowers' aggregate welfare by more than it decreases the lenders' aggregate welfare. Moreover, by assumption, the expected costs of higher inflation are lower than the expected returns from central bank lending. The reason why the borrowers' aggregate welfare increases by more than the lenders' aggregate welfare decreases is as follows. All borrowers which are in the market without intervention gain $r^* - r^V$ with the variable rate tender. All lenders which are in the market without intervention lose $r^* - r^V$ or drop out of the market and lose $r^* - r_t < r^* - r^V$. Since, without intervention, the mass of borrowers equals the mass of lenders at r^*, the gain in the borrowers' aggregate welfare overcompensates the loss in the lenders' aggregate welfare. The aggregate welfare of borrowers and lenders in an interbank market with a variable rate tender is illustrated on the right-hand side of Figure VII.

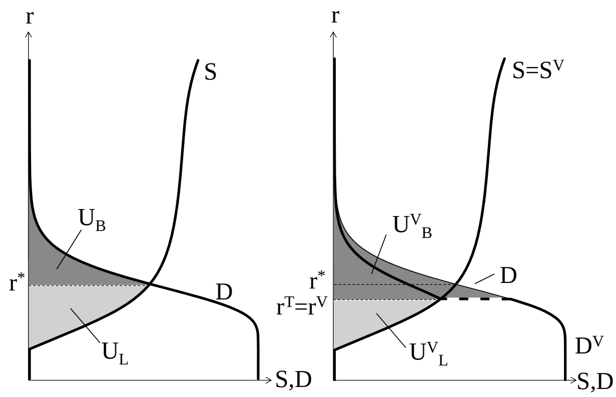

Figure VII: Social welfare in the interbank market without and with a variable rate tender

5.3 Comparison

As we have seen above, both forms of reverse open market operations increase social welfare. If r^I is such that $k = D(r^I) - S(r^I)$, $r^I = r^F = r^V$ and the increase in social welfare is the same for both tenders. However, assume that r^I is chosen such that $k < D(r^I) - S(r^I)$, i.e., $r^I < r^V$. Then, according to Theorem 5, $r^V < r^F$. So, the fixed rate tender's effect on social welfare may differ from the effect of a variable rate tender. In this section, we examine exactly this point.

Theorem 8. Assume $r^V < r^F$. Then, $U_B^F + U_L^F > U_B^V + U_L^V$, whereas $U_C^F < U_C^V$.

Proof. For the proof of Theorem 8, see Appendix A6.

Hence, a fixed rate tender increases the aggregate welfare of borrowers and lenders by more than a variable rate tender. With a fixed rate tender, some of the borrowers only have to pay $r^I < r^V$ such that there is an additional gain in the borrowers' aggregate welfare which is not generated by a variable rate tender. This overcompensates the lower increase in the sum of borrowers' and lenders' aggregate welfare caused by the smaller drop in the interbank rate. However, because $r^I < r^T = r^V$, the central bank faces a lower ratio of the interest rate to the riskiness if it uses a fixed rate tender. Hence, for the central bank, the fixed rate tender is less attractive than the variable rate tender. The relative attractiveness of a fixed rate tender is even lower for low values of r^I. Whether social welfare with a fixed rate tender exceeds social welfare with a variable rate tender ultimately depends on the parameter values.

6 Discussion

In this section, we discuss empirical evidence for our results on the variable rate tender and perform some robustness checks.

6.1 Empirical evidence

The comparison of our results on variable rate tenders with empirical studies not only delivers empirical evidence in favor of our findings. It also supports Ayuso and Repullo's (2003) finding that the bidding behavior of banks is similar in discriminatory (pay what you bid) and uniform-rate auctions. In their comprehensive study, Cassola, Hortaçsu, and Kastl (2011) present a model of bidding in variable rate tenders. In this model, banks with individual liquidity needs can get central bank credits against collateral. They submit a set of interest rate-quantity pairs and pay the submitted interest rates if they are successful with some of their bids. Moreover, secured interbank credits may require collateral of higher quality than secured central bank credits. To show that bids depend both on the willingness to pay for a central bank credit and the strategic response to bids of other banks, Cassola, Hortaçsu, and Kastl (2011) use data on repo auctions of the European Central Bank.[14] For the variable rate tenders before August 9, 2007 (start of the recent financial crisis), they find the following. Bidders have a good idea of the interest rate at which the variable rate tender will clear. Thereby, Cassola, Hortaçsu, and Kastl (2011) use the average number of interest rate-quantity pairs in a bid as a measure of the bidder's uncertainty about the clearing rate, where a low average number indicates low uncertainty. This corresponds to our assumption that bidders anticipate r^T. Cassola, Hortaçsu, and Kastl (2011) find evidence for bid shading, i.e., bids do not correspond with the bidders' individual valuation for a central bank credit. This is in line with our finding that banks do not necessarily bid their reserve rate. In the variable rate tender of the European Central Bank, bids are concentrated on the repo rate, i.e., the interbank rate. This finding corresponds to the equilibrium bidding behavior of borrowers with $q \geq r^V$ described in Lemma 4. Both before and after August 2007, the variable rate tenders of the European Central Bank are efficient in the sense that banks with the highest marginal values for central bank credits are awarded the liquidity. This supports our finding that only borrowers with $q \geq r^V$ may be served with a central bank credit. Moreover, Cassola, Hortaçsu, and Kastl (2011) find that before August 2007, the bidders' valuation for central bank credits is weakly higher than the interbank rate. In our set-up, this would require that, instead of submitting $b_q < r^V$ and being unsuccessful with probability one, borrowers with $q < r^V$ do not participate in the variable rate tender. Imposing this

[14]In particular, they use data on all submitted bids in 50 discriminatory repo auctions of liquidity provided via collateralized loans with 1-week maturity conducted as part of the regular main refinancing operations of the European Central Bank between January 4, 2007 and December 11, 2007.

assumption, however, would not change our results.

6.2 Robustness

In this section we consider some of the assumptions made in Sections 3 and 4 and briefly discuss how changes in these assumptions influence the effect of liquidity injections on the equilibrium interbank rate.

An important assumption is that $pX(p)$ decreases with p, i.e., the higher the probability of success, the lower the expected project return. This assumption is required for adverse selection to emerge. Assume that $pX(p)$ strongly increases with p. This could cause the reserve rate q to increase with p such that borrowers with low probability of default drop out of the interbank market at higher interest rates than borrowers with high probability of default. But then, instead of adverse selection, the interbank market is characterized by favorable selection where an interest rate attracts those borrowers with relatively low default probability. This kind of selection, however, does not seem to apply to interbank markets.

One could also imagine that either L or I or both differ across borrowers. For example, let L decrease with q such that the borrower's demand for credit increases with the probability of default. This causes adverse selection to worsen because at any interest rate, the proportion of demand stemming from borrowers with high default probability increases. In contrast, if L increases with q, the borrower's demand for credit decreases with the probability of default. As a result, adverse selection mitigates.[15] However, such changes in the set-up do not prevent fixed and variable rate tenders from lowering the interbank rate.

It is very interesting to consider changes in the assumption that in case of project failure, the borrowers' repayment to the lender equals zero. Assume that if the project fails, the repayment to the lender is positive. Moreover, assume that borrowers with high default probability repay much less in case of project failure than borrowers with low default probability. The lenders' expected return on a loan may now be such that they prefer lower over high interest rates. For example, supply could be backward bending, i.e., supply increases at low interest rates and decreases at high interest rates. This may not only lead to multiple market-clearing equilibria,

[15] Note that if L is a random draw and independent of q or if L and I simultaneously differ across borrowers, additional assumptions on the combination of L, q, and I are required for adverse selection to sustain.

but it may also change the effect of central bank tenders on the interbank rate significantly. Fixed and variable rate tenders still decrease demand for interbank credits at high interest rates. However, because supply is backward bending, this may now lead to an increase in the equilibrium interbank rate. Thus, the effect of reverse open market operations may be reversed and instead of lowering the interbank rate, these operations may raise the interbank rate.

In our set-up, we assume that the central bank only partially replaces the interbank market, i.e., $k < D(r^I)$ and $k < D(r^V)$, respectively. Consider first the fixed rate tender and assume that $k \geq D(r^I)$. If all borrowers who apply for a central bank credit receive one, demand in the interbank market is zero for $r \geq r^I$ and positive and strictly higher than supply for $r < r^I$. But then, there is no market-clearing equilibrium in the interbank market and the analysis of the effect of a fixed rate tender on the equilibrium interbank rate is no longer feasible. Consider now the variable rate tender and assume $k \geq D(r^V)$. This implies $D^V(r^V) = S(r^V) = 0$ such that there is no trade in the interbank market equilibrium with a variable rate tender. But then, the effect of the variable rate tender on the interbank rate is no longer defined. Moreover, the fact that no interbank credits are traded in equilibrium affects the borrowers' bidding behavior in the variable rate tender. Now, all borrowers willing to pay r^V want to be successful bidders.

One could also assume that, instead of randomly choosing k borrowers and serving each with a central bank credit, the central bank allocates the k units pro rata, according to the ratio of k to the demand for central bank credits.[16] In a fixed rate tender, the fraction of a central bank credit each participating borrower gets then coincides with the probability to get a central bank credit if the central bank chooses randomly. This in turn implies that with pro rata allotment, residual demand and the effect of the fixed rate tender on the interbank market are the same as with random selection. In a variable rate tender, the central bank may decide that in case there are several bids equal to the kth highest bid, it does not choose randomly among the borrowers with these bids to allocate the remaining credits. Instead, the central bank may decide to allocate the remaining units pro rata, according to the ratio of the remaining units to the number of bidders submitting the kth highest

[16] Such an allotment procedure is used, for example, by the European Central Bank in fixed rate tenders, if the aggregate amount of bids exceeds the total amount of liquidity the European Central Bank wants to allot (cf., European Central Bank, 2010).

bid. Consider the equilibrium in Lemma 4, where all borrowers with $q \geq r^V$ submit $b_q = r^V$. Again, the fraction of a central bank credit each successful bidder gets then coincides with the probability to get a central bank credit if the central bank chooses randomly. Hence, with pro rata allotment, residual demand and the effect of a variable rate tender on the interbank market are the same as with random selection.

For the variable rate tender we assume that all successful bidders have to pay r^T. In discriminatory auctions used by the European Central Bank, for example, successful bidders pay their individual bid. As already discussed at the beginning of this section, the introduction of a discriminatory procedure should not change our results for the equilibrium in the variable rate tender.

7 Concluding remarks

This study has dealt with the effect of fixed and variable rate tenders on the unique equilibrium interbank market rate and social welfare.

It suggests that with asymmetric information on the default probability, both fixed and variable rate tenders decrease the interbank rate. Moreover, a central bank's liquidity injection through a variable rate tender may lead to a lower interbank rate than an injection of the same amount of liquidity through a fixed rate tender. However, a fixed rate tender allows to fine-tune the interbank rate. Moreover, both tenders increase social welfare. Thereby, the increase in the central bank's welfare is higher with a variable rate tender, whereas the increase in aggregate welfare of borrowers and lenders is higher with a fixed rate tender.

Our results propose that, besides historical practice and doctrine, the central bank's choice of the type of reverse open market operations should account for the possibility of asymmetric information on the probability of default.

Appendix

Appendix A1: Proof of Lemma 1

Lemma 1. Assume $\partial|\varepsilon^D|/\partial r \geq 0$ and $\varepsilon^g \leq 1$ for $r \geq q_0$. Then, $\partial S(r)/\partial r \geq 0$, and the market-clearing equilibrium is unique.

Proof. For a single market-clearing equilibrium it suffices that demand and supply intersect at most once. By definition, demand equals W for $r \leq q_0$, it is strictly decreasing for $q_0 < r \leq q_1$, and it is zero for interest rates $r \geq q_1$. The average reserve rate is constant for interest rates $r \leq q_0$ and strictly increasing on the interest rate interval $[q_0, q_1]$. Moreover, the average reserve rate is not defined for interest rates $r > q_1$ and by assumption, $\bar{q}(q_1) = q_1$. Thus, supply is strictly increasing for interest rates $r \leq q_0$ and because of $r - t_0 g(\bar{q}(r)) > 0$ at $r = q_1$, supply is positive at $r = q_1$. Supply is not defined for $r > q_1$ such that it cannot cross demand at these interest rates. Hence, for at most one market-clearing equilibrium, it suffices that supply is weakly increasing for interest rates $r \in [q_0, q_1]$. From its definition it follows that supply is weakly increasing if the ratio of the interest rate to the riskiness $r/g(\bar{q}(r))$ is weakly increasing for $r \in [q_0, q_1]$, i.e.,

$$\begin{aligned} & \frac{\partial}{\partial r}\frac{r}{g(\bar{q}(r))} \geq 0 \\ \Leftrightarrow \quad & \bar{q}(r) \geq r\frac{\partial g(\bar{q}(r))}{\partial \bar{q}}\frac{\bar{q}(r)}{g(\bar{q}(r))}\frac{\partial \bar{q}(r)}{\partial r} \\ \Leftrightarrow \quad & \frac{1}{\varepsilon^g} \geq \frac{\partial \bar{q}(r)}{\partial r}\frac{r}{\bar{q}(r)}. \end{aligned} \tag{A.1}$$

Van den Berg (1994) shows that $\partial|\varepsilon^D|/\partial r \geq 0$ suffices for

$$\frac{\partial \log \bar{q}(r)}{\partial \log r} = \frac{\partial \bar{q}(r)}{\partial r}\frac{r}{\bar{q}(r)} \leq 1 \tag{A.2}$$

to be satisfied. Then, the combination of $\varepsilon^g \leq 1$ and $\partial|\varepsilon^D|/\partial r \geq 0$ suffices for (A.1) to be satisfied and the ratio of the interest rate to the riskiness to be weakly increasing. Note that because $\bar{q}(r)$ is increasing in r, ε^g has to be positive. This confirms that $g(\bar{q}(r))$ is strictly increasing in $\bar{q}(r)$. It follows that under a non-decreasing elasticity of demand and a low elasticity of the relative riskiness of a loan, supply $S(r)$ is weakly increasing and positive on the interval $[q_0, q_1]$ and strictly increasing for interest rates $r < q_0$. Hence, $D(r) - S(r)$ is strictly decreasing or negative for all for all $r \geq q_0$ and there is at most one market-clearing equilibrium. $\square$

Appendix A2: Proof of Lemma 2

Lemma 2. Assume $\left|\varepsilon^{D^B}(r)\right| \geq \left|\varepsilon^{D^A}(r)\right|$ for all $r < q_1$. Then $\bar{q}^B(r) \leq \bar{q}^A(r)$ for all $r < q_1$.

Proof. Let the distribution of reserve rates change from F^A to F^B such that $\varepsilon^{D^B}(r) \leq \varepsilon^{D^A}(r)$ for all $r < q_1$, i.e.

$$\frac{\partial \log(1 - F^B(r))}{\partial \log r} \leq \frac{\partial \log(1 - F^A(r))}{\partial \log r}. \tag{A.3}$$

Since both sides are negative, this is nothing else but $\left|\varepsilon^{D^A}(r)\right| \leq \left|\varepsilon^{D^B}(r)\right|$. Because $\partial \log r = (1/r)\partial r$, (A.3) can be rewritten as

$$-\frac{\partial \log(1 - F^A(r))}{\partial r} \leq -\frac{\partial \log(1 - F^B(r))}{\partial r}. \tag{A.4}$$

Integrating (A.4) over the interval $[r', r]$ for $q_0 < r' < r$ yields

$$\log(1 - F^A(r)) - \log(1 - F^A(r')) \leq \log(1 - F^B(r)) - \log(1 - F^B(r')). \tag{A.5}$$

Applying the negative exponential function to both sides of inequality (A.5), we obtain

$$\frac{1 - F^B(r')}{1 - F^B(r)} \leq \frac{1 - F^A(r')}{1 - F^A(r)}. \tag{A.6}$$

By another integration, one finds

$$\frac{1}{1 - F^B(r)} \int_r^{q_1} (1 - F^B(q))dq \leq \frac{1}{1 - F^A(r)} \int_r^{q_1} (1 - F^A(q))dq. \tag{A.7}$$

Partial integration yields

$$\frac{1}{1 - F^B(r)} \int_r^{q_1} q f^B(q)dq \leq \frac{1}{1 - F^A(r)} \int_r^{q_1} q f^A(q)dq, \tag{A.8}$$

which is just $\bar{q}^B(r) \leq \bar{q}^A(r)$. $\square$

Appendix A3: Proof of Proposition 1

Proposition 1. Assume that

1) $W^B\left(1 - F^B(r)\right) \leq W^A\left(1 - F^A(r)\right)$ for all $r \leq q_1$,

2) $f^B(r)/(1 - F^B(r)) \geq f^A(r)/(1 - F^A(r))$ for all $r < q_1$

3) $J^B H^B(t) \geq J^A H^A(t)$ for all t.

Then, $r^B \leq r^A$, where r^i denotes the respective interest rate in the unique market-clearing equilibrium in interbank market $M^i = M^A, M^B$.

Proof. We prove Proposition 1 by contradiction. Assume $r^B > r^A$. Then, from condition 1 in Proposition 1 it follows that $D^B(r^B) \leq D^B(r^A) \leq D^A(r^A)$. On the other hand, $S^B(r^B) \geq S^B(r^A) \geq S^A(r^A)$. The first inequality follows from Lemma 1 and the fact that the elasticity of demand is non-decreasing and the elasticity of relative riskiness is low in both markets. For the second inequality, consider the following. Because of condition 3, it suffices to show that $r/g(\bar{q}^B(r)) \geq r/g(\bar{q}^A(r))$ for $r = r^A$. By Lemma 2, condition 2 is sufficient for this inequality to be satisfied. Because $D^B(r^B) = S^B(r^B)$ and $D^A(r^A) = S^A(r^A)$, $D^B(r^B) \leq D^B(r^A) \leq D^A(r^A) = S^A(r^A) \leq S^B(r^A) \leq S^B(r^B)$. But then, necessarily, all the weak inequalities must be equalities. In particular, $S^B(r^A) = S^B(r^B) = D^B(r^B) = D^B(r^A)$. From the uniqueness of the market-clearing equilibrium follows that $r^B = r^A$, which is the required contradiction. □

Appendix A4: Proof of Theorem 6

Theorem 6. Assume $k \leq D(r^I) - S(r^I)$ and $r^I < r^*$ such that $r^F < r^*$. Then, $U^F > U$.

Proof. The difference between the equilibrium social welfare in the interbank market with a fixed rate tender and without any intervention is given by the difference between (21) and (17), i.e.,

$$x \int_{r^F}^{r^*} D(r)dr + (1-x) \int_{r^I}^{r^*} D(r)dr - \int_{r^F}^{r^*} S(r)dr + k\left(r^I - t_c g(\bar{q}(r^I))\right). \tag{A.9}$$

To prove Theorem 6, it remains to show that (A.9) is positive. Rewriting the first and the second term in (A.9) yields

$$\int_{r^F}^{r^*} D(r)dr + (1-x) \int_{r^I}^{r^F} D(r)dr. \tag{A.10}$$

By Definition 1, $D(r^*) = S(r^*)$ and by Lemma 1, $D(r) > S(r)$ for $r < r^*$. Therefore, $\int_{r^F}^{r^*} D(r)dr > \int_{r^F}^{r^*} S(r)dr$. Moreover, by assumption, $r^I - t_c g(\bar{q}(r^I)) > 0$. Thus, (A.9) is indeed positive. From this it follows that $U^F > U$, i.e., the equilibrium social welfare with a fixed rate tender is higher than the equilibrium social welfare without any intervention. □

Appendix A5: Proof of Theorem 7

Theorem 7. Assume $k < D(r^V)$ and the tender equilibrium to be as described in Lemma 4 such that $r^V < r^*$. Then, $U^V > U$.

Proof. The difference between the equilibrium social welfare in the interbank market with a variable rate tender and without any intervention is given by the difference between (22) and (17), i.e.,

$$\int_{r^V}^{r^*} D(r)dr - \int_{r^V}^{r^*} S(r)dr + k\left(r^V - t_c g(\bar{q}(r^V))\right). \tag{A.11}$$

To prove Theorem 7, it remains to show that (A.11) is positive. Given Lemma 1, $D(r^*) = S(r^*)$ and $D(r) > S(r)$ for $r < r^*$ such that $\int_{r^V}^{r^*} D(r)dr > \int_{r^V}^{r^*} S(r)dr$. Moreover, by assumption, $r^V - t_c g(\bar{q}(r^V))$ is positive. Thus, (A.11) is indeed positive. From this it follows that $U^V > U$, i.e., the equilibrium social welfare with a variable rate tender is higher than the equilibrium social welfare without any intervention. □

Appendix A6: Proof of Theorem 8

Theorem 8. Assume $r^V < r^F$. Then, $U_B^F + U_L^F > U_B^V + U_L^V$, whereas $U_C^F < U_C^V$.

Proof. The difference between U^F and U^V is given by

$$\begin{aligned}
&(U_B^F + U_L^F) - (U_B^V + U_L^V) + (U_C^F - U_C^V) \\
&= (1-x)\int_{r^I}^{r^F} D(r)dr + \int_{r^V}^{r^F} \left(S(r) - D(r)\right)dr \\
&+ k\left[r^I - r^V - t_c\left(g(\bar{q}(r^I)) - g(\bar{q}(r^V))\right)\right].
\end{aligned} \tag{A.12}$$

Note that $S(r) - D(r) < 0$ for $r \in [r^V, r^F]$ and recall that $r^I < r^V$. At interest rate r^I, $(1-x)D(r^I) = k$. Moreover, at interest rate r^V, $D(r^V) - S(r^V) = k$. For $(1-x)\int_{r^I}^{r^F} D(r)dr > \left|\int_{r^V}^{r^F}\left(S(r) - D(r)\right)dr\right|$, i.e., $(U_B^F + U_L^F) > (U_B^V + U_L^V)$, it therefore suffices that in the interval (r^V, r^F), $D(r) - S(r)$ decreases by more than $(1-x)D(r)$ as r increases, i.e.,

$$\frac{\partial}{\partial r}(1-x)D(r) > \frac{\partial}{\partial r}\left(D(r) - S(r)\right). \tag{A.13}$$

The left-hand side of (A.13) is given by $-k/D(r^I) \cdot Wf(r)$. The right-hand side of (A.13) is given by

$$-Wf(r) - Jh\left(\frac{r}{g(\bar{q}(r))}\right)\frac{\partial}{\partial r}\frac{r}{g(\bar{q}(r))}. \tag{A.14}$$

Because $\partial/\partial r(r/g(\bar{q}(r))) > 0$, this derivative is negative. Moreover, because $k/D(r^I) < 1$, $-k/D(r^I) \cdot Wf(r) > -Wf(r)$ and the left-hand side of (A.13) is indeed larger than the right-hand side of (A.13). From this it follows that indeed, $(U_B^F + U_L^F) > (U_B^V + U_L^V)$. From Lemma 1 we know that $r^I/g(\bar{q}(r^I)) < r^V/g(\bar{q}(r^V))$ and from $r^I < r^V$ follows $g(\bar{q}(r^I)) < g(\bar{q}(r^V))$ such that $r^I - t_c g(\bar{q}(r^I)) < r^V - t_c g(\bar{q}(r^V))$. From this it follows that $U_C^F < U_C^V$. $\square$

References

Akerlof, G. (1970): "The Market for "Lemons": Quality Uncertainty and the Market Mechanism," *The Quarterly Journal of Economics*, 84, 488–500.

Allen, F., E. Carletti, and D. Gale (2009): "Interbank Market Liquidity and Central Bank Intervention," *Journal of Monetary Economics*, 56, 639–652.

Ayuso, J., and R. Repullo (2003): "A Model of the Open Market Operations of the European Central Bank," *The Economic Journal*, 113, 883–902.

Bindseil, U. (2002): "Equilibrium Bidding in the Eurosystem's Open Market Operations," *European Central Bank Working paper series 137.*

——— (2004): *Monetary Policy Implementation - Theory, Past, Present.* Oxford University Press Oxford.

Bindseil, U., K. G. Nyborg, and I. Strebulaev (2004): "Bidding and Performance in Repo Auctions: Evidence from ECB Open Market Operations," *CEPR Working paper No 4367.*

Cassola, N., A. Hortaçsu, and J. Kastl (2011): "The 2007 Subprime Market Crisis Through the Lens of European Central Bank Auctions for Short-Term Funds," *Working paper.*

Catalao-Lopes, M. (2010): "Fixed- and Variable-Rate Tenders in the Management of Liquidity by the Eurosystem: Implications of the Recent Credit Crisis," *International Journal of Central Banking*, 6, 199–230.

Chiu, J., and T. V. Koeppl (2011): "Market Freeze and Recovery: Trading Dynamics Under Optimal Intervention by a Market-Maker-of-Last-Resort," *Working paper.*

European Central Bank (2011): "The Implementation of Monetary Policy in the Euro Area," *General Documentation on Eurosystem Monetary Policy Instruments and Procedures.*

Ewerhart, C., and P. Feubli (2012a): "Credit Rationing in the After-Math of the Arnold-Riley Impossibility Theorem," *Working paper.*

——— (2012b): "Monotone Comparative Statics in Markets with Asymmetric Information," *Working paper.*

——— (2012c): "Unique Market-Clearing Equilibrium in Markets for "Lemons"," *Working Paper.*

Guerrieri, V., and R. Shimer (2012): "Dynamic Adverse Selection: A Theory of Illiquidity, Fire Sales, and Flight to Quality," *NBER Working paper 17876.*

Heider, F., M. Hoerova, and C. Holthausen (2009): "Liquidity Hoarding and Interbank Market Spreads - The Role of Counterparty Risk," *European Central Bank Working paper No 1126.*

House, C. L., and Y. Masatlioglu (2010): "Managing Markets for Toxic Assets," *NBER Working paper 16145.*

Kharroubi, E., and E. Vidon (2009): "Liquidity, Moral Hazard, and Interbank Market Collapse," *International Journal of Central Banking*, 5, 51–86.

Krishna, V. (2010): *Auction Theory.* Elsevier Amsterdam, second edn.

McAndrews, J., A. Sarkar, and Z. Wang (2008): "The Effect of the Term Auction Facility on the London Inter-Bank Offered Rate," Staff Report 335, Federal Reserve Bank of New York.

Nautz, D., and J. Oechssler (2003): "The Repo Auctions of the European Central Bank and the Vanishing Quota Puzzle," *Scandinavian Journal of Economics*, 105, 207–220.

Philippon, T., and V. Skreta (2011): "Optimal Interventions in Markets with Adverse Selection," *Working paper.*

Taylor, J. B., and J. C. Williams (2009): "A Black Swan in the Money Market," *American Economic Journal: Macroeconomics*, 1, 58–83.

Tirole, J. (2011): "Overcoming Adverse Selection: How Public Intervention Can Restore Market Functioning," *American Economic Review*, forthcoming.

Van den Berg, G. (1994): "The Effects of Changes of the Job Offer Arrival Rate on the Duration of Unemployment," *Journal of Labor Economics*, 12, 478–498.

Wilson, C. (1980): "The Nature of Equilibrium in Markets with Adverse Selection," *The Bell Journal of Economics*, 11, 108–130.

Wu, T. (2011): "The U.S. Money Market and the Term Auction Facility in the Financial Crisis of 2007-2009," *The Review of Economics and Statistics*, 93, 617–631.